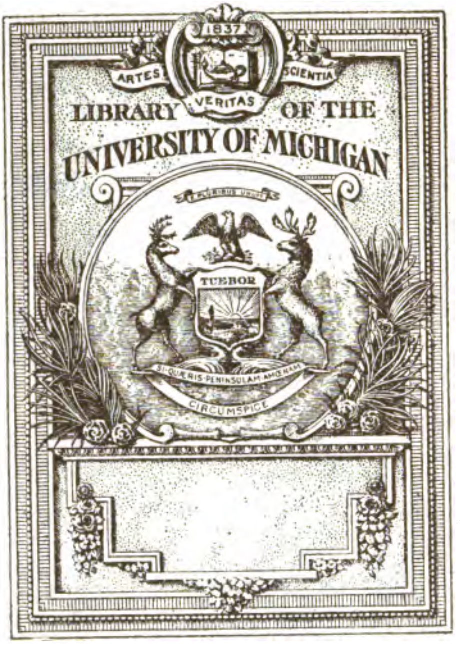
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ON

ROULETTES AND GLISSETTES.

BY

W. H. BESANT, D.Sc., F.R.S.,
William Henry

FELLOW OF ST JOHN'S COLLEGE, CAMBRIDGE.

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PREFACE TO THE FIRST EDITION.

THE following pages contain the explanation of methods, and the investigation of formulæ, which I have for some time past found useful in the discussion of the curves produced by the rolling or sliding of one curve on another.

These methods and formulæ are with a few exceptions original, and, I believe, new; and my object has been to present, from a geometrical point of view, solutions of the various problems connected with Roulettes and Glissettes. I have ventured to introduce, and employ, the word Glissette, as being co-expressive with Roulette, a word which has been in use amongst mathematicians for a considerable time.

The formula of Art. (34) is of course well known; it is given in Salmon's *Higher Plane Curves*, in Walton and Campion's *Solutions*, in Jullien's *Problems*, in Bertrand's *Differential Calculus*, and probably in many other books.

The theorem of Art. (37) was enunciated some years ago, for the particular case of a conic, by Mr Wolstenholme, and extended by myself to the case of any curve. I have however recently found a reference to it in the *Nouvelles Annales* for June, 1869, from which it appears that it was given by Steiner in an early number of the same journal.

For the incisive method of Art. (78) I am indebted to Mr Ferrers.

It will be seen that the general formula of Art. (50) includes most of those which precede it, while it is itself included in that of Art. (77), and that the theorem of Art. (60) reduces all cases of motion in one plane to the cases of Articles (50) or (77).

In a future tract I hope to produce some further developments of the ideas which are here somewhat briefly treated.

W. H. BESANT.

December, 1869.

PREFACE TO THE SECOND EDITION.

THESE 'Notes' have been out of print for a long time, and I have frequently been requested to produce a new edition, but, until recently, I have not been able to find the requisite time for the purpose of doing so.

I have made considerable additions to the text and the examples, but, as these notes by no means constitute an exhaustive treatise on the subject, I retain the original title.

I am much indebted to Mr A. W. Flux, Fellow of St John's College, for kind assistance in the revision of proof sheets.

W. H. BESANT.

April, 1890.

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PRELIMINARY REMARKS ON

INFINITESIMALS.

1. An infinitesimal is a quantity which, under certain assigned conditions, vanishes compared with finite quantities.

If two infinitesimals vanish in a finite ratio to each other, they are said to be of the same order.

Thus, if θ vanish, $\sin\theta$ and θ are of the same order, as are also $\sin m\theta$ and $\tan n\theta$.

If two infinitesimals, α and β, are such that the ultimate ratio of β to α^2 is finite, β is said to be of the second order if α be of the first order.

Thus, $1-\cos\theta$, when θ vanishes, is of the second order if θ be of the first order.

And, generally, an infinitesimal which has, ultimately, a finite ratio to the rth power of another is said to be of the rth order if that other be of the first order.

The order of an infinitesimal is, *à priori*, arbitrary and conventional; but, if any standard be fixed upon, the orders of all others are determinate.

2. Consider figure (1), in which O is the centre of a circle, and AP a small arc; PN, PL perpendiculars on OA and on the tangent at A, and Q the point in which OP produced meets AL.

Then, if $OA = a$, and $AOP = \theta$, it can be shewn by Trigonometry that, when θ is indefinitely diminished,

$$\frac{AL}{AP} = 1, \quad \frac{PL}{AP^2} = \frac{1}{2a}, \quad \frac{QL}{AP^3} = \frac{1}{2a^2},$$

and
$$\frac{PQ - PL}{AP^4} = \frac{1}{4a^3}.$$

Therefore, if AP is an infinitesimal of the first order, AL is of the first order, PL of the second, QL of the third, and $PQ - PL$ of the fourth.

3. If a is an infinitesimal of the first order,

$$\lambda a^2 + \mu a^3 : \nu a^2 :: \lambda : \nu, \text{ ultimately,}$$

$$\lambda a^2 + \mu a^3 \text{ is of the second order:}$$

and generally, it will be seen that the order of an infinitesimal is not affected by the addition to it of an infinitesimal of any higher order.

If AP' is a small arc of a curve, and AP an equal arc of its circle of curvature at A; then PP' is of the third order, and therefore, so far as quantities of the second order are concerned, P' may be taken to be coincident with P.

4. If AP, AQ are two infinitesimal arcs, of the first order, of two curves touching each other at A, and if AP, AQ be equal, or ultimately equal to each other, the distance PQ is of the second order, and therefore, so far as quantities of the first order are concerned, P and Q may be taken to be coincident.

It will be seen that all the preceding theorems are contained in, or deducible from, the 7th and 11th lemmas of the first section of the *Principia*.

Thus, from Lemma XI., if AP, AP' are two infinitesimal arcs of a curve of the same order, and PL, $P'L'$ the corresponding perpendiculars,

$$PL : P'L' :: AP^2 : AP'^2.$$

ROULETTES.

5. WHEN a curve rolls on a fixed curve any given point in the plane of the rolling curve describes a certain curve, which is called a roulette.

Under the same heading we shall also include the curves enveloped by any given lines, straight or curved, which are carried with the rolling curve.

In dealing with roulettes the following is a fundamental theorem.

If a curve roll on a fixed curve, the line joining the point of contact with any point Q *in the plane of the rolling curve is the normal to the path of* Q.

For, as the curve rolls, the point of the curve, P, in contact with the fixed curve, has no motion, and the whole area is, at the instant, turning round P: hence the direction of motion of Q, i.e. the tangent to its path, is at right angles to QP, and QP is the normal. (See fig. 2.)

6. *Centre of curvature of roulette.*

If PP', Pp are equal infinitesimal arcs of the fixed and rolling curves, so that Qp rolls into the position $Q'P'$ (as in fig. 9), QP, $Q'P'$ are consecutive normals of the roulette, and E, the point of intersection of these lines, produced if necessary, is the centre of curvature at the point of the roulette.

7. We commence with two particular cases as illustrations.

If a circle roll on the inside of the circumference of a circle of double its radius, any point in the area of the rolling circle traces out an ellipse.*

Let C be the centre of the rolling circle, and E the point of contact (fig. 3).

Then, if the circle meet in Q, a fixed radius of the fixed circle, the angle ECQ is twice the angle EOA, and therefore the arcs EQ, EA are equal.

Hence when the circles touch at A, the point Q of the rolling circle coincides with A, and the subsequent path of Q is the diameter through A.

Let P be a given point in the given radius CQ, and draw RPN perpendicular to OA;

then, OQE being a right angle, EQ is parallel to RP, and therefore $CR = CP$, and OR is constant.

Also $$PN : RN :: PQ : OR;$$

therefore the locus of R being a circle, the locus of P is an ellipse, whose semi-axes are

$$OC + CP \text{ and } OC - CP.$$

8. Properties of the ellipse are deducible from this construction.

Thus, the point E being the instantaneous centre, PE is the normal to the ellipse at the point, and PT, perpendicular to it, and therefore parallel to OF, is the tangent.

A circle can be drawn through $EPQT$, since EPT, EQT are right angles; but the circle through QPE clearly passes through R,
therefore, the angle ORT is a right angle and

$$ON : OR :: OR : OT$$

or $$ON.OT = OR^2,$$

a known property of the tangent.

* Appendix to *Geometrical Conics*, first edition, 1869.

Again, if PF meet OQ in G, the angles PQG, PFQ are equal, being on equal bases, EQ, OQ';

$$\therefore\ PG : PQ :: PQ : PF$$

or
$$PG \,.\, PF = PQ^2 = OR'^2,$$

a known property of the normal.

9. *To find the intrinsic equation of a cycloid.*

If the circle BPQ rolls along the straight line AP, the diameter BQ originally coinciding with AO (fig. 4), the point Q traces out a cycloid of which O is the vertex.

QP is the normal at the point Q of the cycloid, and if Pp is an elementary arc of the circle, Qp turns into the position $Q'P'$, so that $Q'P'$ is the consecutive normal, and the point E is the centre of curvature.

$P'p$ being of the second order of infinitesimals, the points p and P' may be taken to be coincident, and if $PCp = \delta\theta$, $PQp = \frac{1}{2}\delta\theta$.

As the circle turns through the angle $\delta\theta$, Qp turns through the same angle, and therefore $QpQ' = \delta\theta$; hence it follows that $QEQ' = \frac{1}{2}\delta\theta$, and therefore, ultimately, $QE = 2QP$.

If the arc $QQ' = \delta s$, we have

$$\delta s = 2PQ \,.\, \tfrac{1}{2}\delta\theta = 2a\cos\frac{\theta}{2}\,\delta\theta,$$

and, QR being the tangent Q, the angle ϕ of deflection

$$= RPQ = \tfrac{1}{2}\theta;$$

$$\therefore\ \delta s = 4a\cos\phi\,\delta\phi$$

and
$$s = 4a\sin\phi,$$

measuring s and ϕ from the point O, and the tangent at O.

10. *A curve rolls on a straight line; it is required to find the roulette traced by any point* Q.

Let the curve roll from O to P, the point A passing over the point O. (See fig. 2.)

Taking O as the origin, and OP as axis of x, let x, y, be co-ordinates of Q.

Then, if $$AQP = \theta, \quad QP = r,$$

$$\tan QPN = \frac{dx}{dy} = r\frac{d\theta}{dr},$$

and $$y = r \sin QPN = r\frac{dx}{ds}.$$

Hence, if the polar equation referred to the point Q, $r = f(\theta)$, is given, we have three equations from which r and θ can be eliminated, and the resulting equation will be the differential equation to the path of Q.

Or, if the arc AP $(= s)$ be found in terms of θ, we may employ the equations

$$x = s - r\cos NPQ = s - r\frac{dr}{ds},$$

$$y = r\sin QPN = r^2\frac{d\theta}{ds},$$

and the elimination of r and θ will give the equation in x and y, to the path of Q.

If the rolling curve is given by the equation $p = f(r)$, we have

$$p = QN = r\frac{dx}{ds};$$

$$\therefore \text{ since } p = y,$$

$$y = f\left(y\frac{ds}{dx}\right)$$

is the roulette.

11. The two theorems following will be found of great use in the discussion of roulettes.

If ϕ is the deflection of the tangent at any point P of a curve from the tangent at a fixed point of the curve from which the arc is measured, and if p is the perpendicular from a fixed point (O) on the tangent at P, then

(1) *The perpendicular from* O *on the normal, measured in the same direction as the arc, is equal to* $\frac{\mathrm{dp}}{\mathrm{d}\phi}$ *if the curve is concave to the point* O, *and is* $-\frac{\mathrm{dp}}{\mathrm{d}\phi}$ *if the curve is convex to the point* O.

(2) *The radius of curvature at* $\mathrm{P} = \mathrm{p} + \frac{\mathrm{d}^2\mathrm{p}}{\mathrm{d}\phi^2}$, *if the curve is concave to the point* O, *and* $= -\mathrm{p} - \frac{\mathrm{d}^2\mathrm{p}}{\mathrm{d}\phi^2}$, *if the curve is convex to the point* O.

In fig. 5, if PP' is an elemental arc of the curve, and if the tangents at P and P' intersect in T, $Y'TY = \delta\phi$, and $KY' = TK\delta\phi$, neglecting infinitesimals of the second order.

But $\delta p = OY' - OY = KY'$ to the first order;

therefore ultimately $\qquad dp = PYd\phi,$

or
$$OZ = \frac{dp}{d\phi},$$

when measured in the direction PP'.

In fig. 6,

$$\delta p = OY' - OY = -KY = -TK\delta\phi,$$

so that $OZ = -\frac{dp}{d\phi}$, when measured in the direction $P'P$, and is therefore $\frac{dp}{d\phi}$ in the direction PP'.

In fig. 7,

$$\delta p = OY' - OY = -KY = -TK\delta\phi,$$

so that
$$OZ = -\frac{dp}{d\phi},$$

measured in the direction PP'.

And in fig. 8,

$$\delta p = KY' = TK\delta\phi,$$

therefore $OZ = \frac{dp}{d\phi}$, measured in the direction $P'P$, and this $-\frac{dp}{d\phi}$ in the direction PP'.

Again, in fig. 5,

$$\delta s = PT + TP' = TY - PY + P'Y' - TY'$$

$$= \delta \,.\, PY + KY = \delta \,.\, \frac{dp}{d\phi} + p\delta\phi = \left(\frac{d^2p}{d\phi^2} + p\right)\delta\phi,$$

and radius of curvature $= \frac{ds}{d\phi} = p + \frac{d^2p}{d\phi^2}$.

In fig. 6, $\delta s = PY - TY + TY' - P'Y' = -\delta \,.\, PY + KY'$

$$= \frac{d^2p}{d\phi^2}\delta\phi + p\delta\phi,$$

$$\therefore \frac{ds}{d\phi} = p + \frac{d^2p}{d\phi^2}.$$

In fig. 7, $\delta s = TY - PY + P'Y' - TY' = \delta \,.\, PY - KY'$

$$= -\frac{d^2p}{d\phi^2}\delta\phi - p\delta\phi;$$

so that $\qquad \frac{ds}{d\phi} = -p - \frac{d^2p}{d\phi^2};$

and in fig. 8,

$$\delta s = PY - TY + TY' - P'Y' = -\delta \,.\, PY - KY$$

$$= -\frac{d^2p}{d\phi^2}\delta\phi - p\delta\phi;$$

so that $\qquad \frac{ds}{d\phi} = -p - \frac{d^2p}{d\phi^2}.$

The cases of figures 5 and 7 are sufficient for the argument; the other cases are given for fullness of illustration. The same results are obtained by analytical methods, as in Todhunter's *Integral Calculus*, Art. 90.

12. If ϕ is the inclination of the tangent, or the normal, to any fixed direction, and if p is the perpendicular from a fixed point on the tangent to a curve, the relation, $p = f(\phi)$, is called the tangential polar equation of the curve*.

We may remark that if ϕ is the inclination of the normal to a fixed line, p and ϕ are the polar coordinates of the point Y; so that, putting r and θ for p and ϕ, the polar equation of the pedal curve is

$$r = f(\theta).$$

For instance, the polar equation of the pedal of an ellipse with regard to its centre is

$$r^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta;$$

so that the tangential polar equation of an ellipse, referred to its centre, is

$$p^2 = a^2 \cos^2 \phi + b^2 \sin^2 \phi.$$

13. We may observe that the intrinsic equation is at once obtained by integrating the equation

$$\frac{ds}{d\phi} = p + \frac{d^2p}{d\phi^2},$$

the right-hand member being expressed in terms of ϕ.

For instance, if OY is the perpendicular from O on the tangent RQ of a cycloid, (fig. 4), $OY = OR \sin \phi$ and therefore $p = 2a\phi \sin \phi$ is the tangential polar equation of a cycloid referred to its vertex.

Hence $\dfrac{ds}{d\phi} = 4a \cos \phi$, and $s = 4a \sin \phi$.

14. *To find the tangential polar equation of the roulette traced by a point.*

Let fall OY, OZ, perpendiculars on the tangent QT and the normal QP (fig. 10).

* This title was suggested by Dr Ferrers. (*Cambridge and Dublin Mathematical Journal*, 1855.)

Let $OY = p$, and $YOT = \phi$.

Then
$$p = r - OP \cos\phi,$$
$$= r - s\cos\phi,$$
whence, having r and s in terms of θ, and $\tan\phi$ being $\dfrac{rd\theta}{dr}$, we can eliminate r and θ and get the relation between p and ϕ.

Or, without finding the arc, we have
$$\frac{dp}{d\phi} = OZ = PZ\tan\phi$$
$$= (r - p)\tan\phi,$$
and, eliminating θ between this equation and
$$\tan\phi = r\frac{d\theta}{dr},$$
we get the differential equation, in p and ϕ, of the roulette.

15. Ex. 1. *To find the roulette traced by the focus of a parabola rolling on a straight line.*

In this case
$$\frac{2a}{r} = 1 + \cos\theta,$$
the point A (fig. 2), being the vertex of the curve, and Q the focus;
$$\therefore \tan QPN = \cot\frac{\theta}{2} = \frac{dx}{dy},$$
and
$$y = r\sin QPN = \frac{a}{\cos\frac{\theta}{2}};$$
$$\therefore \left(\frac{dx}{dy}\right)^2 = \frac{a^2}{y^2 - a^2},$$
whence, by integration,
$$y = \frac{a}{2}\left(\epsilon^{\frac{x}{a}} + \epsilon^{-\frac{x}{a}}\right).$$

That is, the roulette is a catenary.

Ex. 2. *The curve* $r = a$ *versin* θ *rolls on its axis; required the locus of its pole.*

Taking the second system of equations of Art. 10, and observing that

$$QPN = \frac{\theta}{2}, \text{ and } s = 4a\left(1 - \cos\frac{\theta}{2}\right);$$

we find that

$$x = 4a\left(1 - \cos\frac{\theta}{2}\right) - a\,(1 - \cos\theta)\cos\frac{\theta}{2}$$

$$= 4a - 2a\cos\frac{\theta}{2}\left(2 + \sin^2\frac{\theta}{2}\right),$$

and $$y = 2a\sin^3\frac{\theta}{2};$$

whence $$4a - x = \sqrt{(2a)^{\frac{2}{3}} - y^{\frac{2}{3}}}\,\{2\,(2a)^{\frac{2}{3}} + y^{\frac{2}{3}}\}.$$

16. If the roulette be given in terms of x and y, we can at once find the rolling curve.

For, Art. 10,

$$p = y, \text{ and } p = r\frac{dx}{ds}.$$

Hence, if $y = f(x)$ be the roulette, we can eliminate x and y, and find the equation, in p and r, to the rolling curve, referred to the carried point as origin.

Ex. 1. If the roulette be the catenary

$$y = \frac{c}{2}\left(\epsilon^{\frac{x}{c}} + \epsilon^{-\frac{x}{c}}\right),$$

$$\frac{dx}{ds} = \frac{c}{y}, \text{ and } p = r\,.\,\frac{c}{y} = \frac{rc}{p},$$

$$\therefore\ p^2 = rc,$$

that is, the rolling curve is a parabola.

Ex. 2. If the roulette be

$$y^2 = 4ax,$$

$$\left(\frac{r}{p}\right)^2 = \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{4a^2}{y^2};$$

$$\therefore\ p^2 = r^2 - 4a^2,$$

the involute of a circle.

Ex. 3. If the roulette be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the curve is

$$r^2 = e^2p^2 + \frac{b^4}{a^2},$$

an epicycloid (Art. 22).

Ex. 4. If the roulette be

$$\frac{ds}{dx} = f(y),$$

the curve is

$$r = pf(p).$$

17. *Roulettes of circles.*

The equations in x and y, or in polar coordinates, of the roulettes produced by circles rolling on straight lines or on circles are at once obtained from figures. Thus, in the case of the cycloid, if $ON = x$, and $PN = y$ (fig. 4),

$$x = a - a\cos\theta, \quad y = a\theta + a\sin\theta,$$

and therefore $$y = a\,\text{vers}^{-1}\frac{x}{a} + \sqrt{2ax - x^2},$$

is the Cartesian equation of a cycloid.

Again a cardioid is the roulette produced by a point in the circumference of a circle rolling on an equal circle, and if P, the tracing point, starts from A (fig. 11), and if $ON = x$, and $PN = y$, the cardioid is given by the equations

$$x = 2a\cos\theta - a\cos 2\theta, \quad y = 2a\sin\theta - a\sin 2\theta.$$

If $AP = r$, we at once get the polar equation of the cardioid referred to its cusp,

$$r = 2a(1 - \cos\theta),$$

PQ being the diameter through P, the point Q traces out a cardioid of which B is the cusp, and if $BQ = r$, its equation is

$$r = 2a(1 + \cos\theta).$$

Again, since

$$BF = BE + EF = 2a\cos\theta + 2a(1 - \cos\theta) = 2a,$$

the locus of F is a circle, centre B and radius AB, and FP is the tangent at F.

Hence it follows that the cardioid described by the point P is the pedal with regard to A, of the circle, centre B and radius BA.

18. *Epicycloids and Hypocycloids.*

An epicycloid is the curve traced by a point in the circumference of a circle rolling outside a fixed circle.

A hypocycloid is the curve traced by a point in the circumference of a circle rolling inside a fixed circle.

Thus for an epicycloid, if a and b are the radii of the fixed and rolling circles (fig. 12), and if

$$AOP = \theta, \quad QCP = \frac{a\theta}{b};$$

$$\therefore\ x = (a + b)\cos\theta - b\cos\frac{a+b}{b}\theta,$$

$$y = (a + b)\sin\theta - b\sin\frac{a+b}{b}\theta.$$

For a hypocycloid we obtain in the same manner

$$x = (a - b)\cos\theta + b\cos\frac{a-b}{b}\theta,$$

$$y = (a - b)\sin\theta - b\sin\frac{a-b}{b}\theta.$$

19. The area swept over by the radius vector is most easily found by help of these equations and the expression

$$\frac{1}{2}\int(xdy - ydx),$$

which is at once obtained from figure (13) as follows.

If OP, OQ are consecutive radii of a curve, x, y the coordinates of P, and $x + dx$, $y + dy$, of Q, the elemental triangle

$$\begin{aligned} OPQ &= OQL - OPN - PL - PQR \\ &= \frac{1}{2}(x + dx)(y + dy) - \frac{1}{2}xy - ydx - \frac{1}{2}dx\,.\,dy \\ &= \frac{1}{2}(xdy - ydx), \end{aligned}$$

and therefore the area swept over by the radius vector

$$= \frac{1}{2}\int(xdy - ydx).$$

For instance, in the case of a cycloid (fig. 4), the area swept over by OQ from the vertex to the cusp

$$\begin{aligned} &= \frac{1}{2}\int(QN\,.\,d\,.\,ON - ON\,.\,d\,.\,QN) \\ &= \frac{a^2}{2}\int_0^\pi\{(\theta + \sin\theta)\sin\theta - (1 - \cos\theta)(1 + \cos\theta)\}\,d\theta \\ &= \frac{1}{2}\pi a^2. \end{aligned}$$

Adding πa^2, and doubling the result, we obtain the area between the curve and the straight line joining two consecutive cusps.

20. The roulettes traced by the centres of circles rolling on curves belong to the class of *parallel curves*.

If b is the radius of the circle, and if x, y are the coordinates of its centre, and x', y' of the point of contact,

$$x = x' + b\cos\phi \quad \text{and} \quad y = y' + b\sin\phi,$$

where ϕ is the inclination of the normal to the axis of x.

Or, if the given curve be $p = f(\phi)$, the parallel is

$$p = f(\phi) + b.$$

Thus the parallels of a parabola referred to its focus and of an ellipse referred to its centre are respectively

$$p = a \sec \phi + d, \text{ and } p = \sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi} + d.$$

21. *To find the tangential polar equation, and the intrinsic equation, of an epicycloid.*

In fig. (12), let $OA = a$, $CP = b$, and $AOP = \theta$.

Then, if Q is the point tracing out the epicycloid, QD is the tangent at Q, and taking ϕ as the deflection of the tangent at Q from the tangent at A, ϕ is the inclination of OY the perpendicular p on the tangent at Q, to the fixed line OB at right angles to OA.

From the figure,

$$\phi - \theta = CDQ = \frac{1}{2} PCQ = \frac{a\theta}{2b},$$

and $\therefore$

$$p = (a + 2b) \sin \frac{a\theta}{2b}.$$

But

$$\phi = \frac{a + 2b}{2b} \theta,$$

$$\therefore p = (a + 2b) \sin \frac{a\phi}{a + 2b},$$

is the tangential polar equation of the epicycloid.

Hence

$$\frac{ds}{d\phi} = (a + 2b) \left(1 - \frac{a^2}{(a + 2b)^2}\right) \sin \frac{a\phi}{a + 2b}$$

$$= \frac{4b(a + b)}{a + 2b} \sin \frac{a\phi}{a + 2b'},$$

and therefore

$$s = \frac{4b}{a}(a + b)\left(1 - \cos \frac{a\phi}{a + 2b}\right),$$

measuring the arc and the deflection from the cusp A and the tangent at the cusp.

Hence $s : 2 \,\text{vers}.\text{arc}\, PQ :: 2.OC : OP$,

the form given in the *Principia*, section X., prop. XLVIII.

We may observe that the radius of curvature of the epicycloid at Q

$$=\frac{ds}{d\phi}=\frac{4b(a+b)}{a+2b}\sin PDQ=\frac{2(a+b)}{a+2b}PQ.$$

22. To find the equation, in p and r, of an epicycloid.

If $OQ=r$, we have, from the triangle OCQ,

$$r^2=(a+b)^2+b^2-2b(a+b)\cos\frac{a\theta}{b}$$

$$=a^2+4b(a+b)\sin^2\frac{a\theta}{2b}$$

$$=a^2+4b(a+b)\frac{p^2}{(a+2b)^2}.$$

and therefore $p^2=\dfrac{(a+2b)^2}{4b(a+b)}(r^2-a^2)$.

23. *The hypocycloid.*

If a circle of radius b roll inside, or, more generally, with its concavity in the same direction, on a fixed circle of radius a, and if (fig. 14) Q is the tracing point, QE is the tangent at Q to the hypocycloid.

Supposing Q originally at A, so that OA is the tangent at the cusp A, let ϕ be the deflection of the tangent at Q from the tangent at A.

Then, if $AOC=\theta$,

$$p=OY=(a-2b)\sin CEQ=(a-2b)\sin\frac{a\theta}{2b};$$

but $\dfrac{a\theta}{2b}=CEQ=\theta+\phi, \quad \therefore \theta=\dfrac{2b}{a-2b}\phi,$

and $p=(a-2b)\sin\dfrac{a\phi}{a-2b}.$

Since the curve is convex to the point O,

$$\frac{ds}{d\phi} = -p - \frac{d^2p}{d\phi^2} = \frac{4b(a-b)}{a-2b} \sin \frac{a\phi}{a-2b}\,^*,$$

and
$$s = \frac{4b(a-b)}{a-2b}\left(1 - \cos \frac{a\phi}{a-2b}\right).$$

If $b > a$, this may be written in the form

$$\frac{ds}{d\phi} = -4(b-a)\frac{a+(b-a)}{a+2(b-a)} \sin \frac{a\phi}{a+2(b-a)},$$

so that, when $b > a$, the hypocycloid is identical with the epicycloid generated by a circle of radius $b-a$ rolling outside a circle of radius a.

This can also be seen by direct geometry; for if PQ meet the fixed circle in R (fig. 15), let OR produced meet DQ produced in E; then RE is the diameter of a circle, touching at R and passing through Q.

The angle $FQR = FRQ = RPO$; therefore FQ is parallel to OP; and

$$\angle REQ = \frac{\pi}{2} - ERQ = PDQ;$$

$$\therefore RE = OE - OR = 2(b-a),$$

and $OF = b = CQ$, so that OF is parallel to CQ.

Hence $\quad \text{arc}\, RQ = (b-a).POR = (b-a)\,PCQ$

$$= (b-a).\frac{a\theta}{b} = \text{arc}\, RA,$$

so that the point Q, carried by the circle F, will produce the hypocycloid.

24. It may be useful to give the several equations for the case of a three-cusped hypocycloid, or tricusp, a curve possessing many remarkable properties.

* The p and r equation is $p^2 = \frac{(a-2b)^2}{4b(a-b)}(a^2 - r^2)$, and, if $2b = a - c$, this becomes $p^2 = \frac{c^2}{a^2 - c^2}(a^2 - r^2)$, the form given by the Jesuit Fathers in the notes to Prop. LI. of the *Principia*.

If $AOP = \theta$, and if ϕ is the angle of deflection from OA, the tangent at A, of EQ the tangent at Q (fig. 14),

$$QCP = 3\theta, \text{ and } \phi = PEQ - POA = \frac{3\theta}{2} - \theta = \frac{\theta}{2}.$$

Taking $3c$ as the radius of the fixed circle,

$$p = OY = OE \sin CEQ = c \sin \frac{3\theta}{2},$$

$$\therefore\ p = c \sin 3\phi,$$

is the tangential polar equation of the tricusp, referred to its centre.

The curve being convex to the point O,

$$\frac{ds}{d\phi} = -p - \frac{d^2p}{d\phi^2} = 8c \sin 3\phi,$$

$$\therefore\ s = \tfrac{8}{3}c\,(1 - \cos 3\phi),$$

is the intrinsic equation of the tricusp, measuring s and ϕ from a cusp and the tangent at the cusp.

Writing $\frac{\pi}{6} + \psi$ for ϕ, and assuming that s and ψ vanish together, we obtain

$$s = \tfrac{8}{3}c \sin 3\psi,$$

which is the intrinsic equation of the tricusp referred to the middle point between two consecutive cusps, and the tangent at that point.

Again, from the triangle OCQ, if $OQ = r$,

$$r^2 = 4c^2 + c^2 + 4c^2 \cos 3\theta = 9c^2 - 8c^2 \sin^2 \frac{3\theta}{2},$$

$$\therefore\ r^2 = 9c^2 - 8p^2,$$

is the equation, in p and r, of the tricusp.

25. *Properties of the tricusp.*

(1) *The portion of the tangent within the curve is of constant length, and the locus of its middle point is the circle inscribed in the curve.*

Let A be a cusp of the tricusp described by the point P (fig. 16).

If EQ is parallel to DP, the point Q is a point on the tricusp.

For, since the inclination of DP to OA is 2θ, it follows that DP turns round twice as fast as OD, and in the contrary direction, and therefore that Q is the position of P when OD has turned through two right angles.

PQ is equal to DE and is therefore of constant length, and if OBT is parallel to DP, B is the middle point of PQ, and the locus of B is the circle centre O and radius OB.

Now describe the circle, diameter BT, and draw TC perpendicular to PQ.

Then $$CFB = 2\left(\frac{\pi}{2} - OBP\right) = \pi - 2(\pi - 3\theta) = 6\theta - \pi.$$

$$\therefore \text{arc } TBC = 6c\theta = \text{arc } TA, \text{ since } AOB = 2\theta.$$

$\therefore$ the point C is a point on the tricusp, and PQ is the tangent at C.

(2) *The distance between the centres of curvature corresponding to the intersections of the tangent with the curve is constant.*

If K is the centre of curvature at P, in PR produced,

$$PK = 8c \sin 3\phi = 8c \sin \frac{3\theta}{2} = 4PR.$$

and, if K' is the centre of curvature at Q, $QK' = 4QS$.

Hence it follows that KK' is parallel to PQ, and that

$$KK' = PQ + 4SL = 6SL,$$

so that KK' is double the diameter of the fixed circle.

26. (3) *The envelope of the pedal line of a triangle is a tricusp.*

(*Quarterly Journal of Pure and Applied Mathematics*, No. 38, 1869.)

Taking any point P in the circumscribing circle, centre

O, let PK, PL be the perpendiculars on the sides AC, AB, and NY the perpendicular from N, the middle point of AC, on KL, which is called the 'pedal line.'

Then if $ONY = \phi$, and $COP = \theta$ (fig. 17),

$$\frac{\pi}{2} - \phi = LKP = LAP = A - \tfrac{1}{2}\theta,$$

and
$$\begin{aligned} NY &= NK \sin \phi \\ &= R \sin \phi \sin (B + 2A + 2\phi - \pi), \text{ if } OP = R, \\ &= \frac{R}{2}[\cos \{\phi - (C - A)\} - \cos \{3\phi - (C - A)\}]. \end{aligned}$$

Now, if O' be the centre of the nine-point circle,

$$ONO' = C - A,$$

and therefore, taking p as the perpendicular from O' on LK,

$$\begin{aligned} p &= -\frac{R}{2} \cos \{3\phi - (C - A)\} \\ &= -\frac{R}{2} \cos 3\phi', \end{aligned}$$

changing the initial line.

This is the 'tangential polar' equation of a three-cusped hypocycloid, generated by a circle of radius $\frac{R}{2}$ rolling inside a circle of radius $\frac{3R}{2}$.

Hence the envelope of the pedal line of any triangle is a three-cusped hypocycloid, the centre of which is the centre of the nine-point circle.

We may remark that if KP be produced to meet the circle in p, the line Bp is parallel to KL; a simple method is thus found of constructing the various positions of KL.

The question considered is a particular case of the following problem:

Perpendiculars PK, PL are let fall on two fixed straight lines OA, OB; given the locus of P, it is required to find the envelope of KL.

To do this, let p, the perpendicular from O on KL, make with OA the angle ϕ, and let $POA = \theta$, $AOB = \alpha$.

Then $$p = OK \cos\phi = OP \cos\theta \cos\phi;$$
and $$\phi = LKP = LOP = \alpha - \theta;$$
therefore if $$OP = f(\theta),$$
$$p = f(\alpha - \phi)\cos(\alpha - \phi)\cos\phi$$
is the equation to the envelope.

The problem of the 'pedal line' has been discussed in this *Journal*, by Messrs. Greer, Walton, Ferrers, and Griffith; it was, I think, first pointed out by Mr. Ferrers that the centre of the tricusp is the centre of the nine-point circle.

27. It may be noticed that the trilinear equation of the tricusp, referred to the triangle formed by the three cusps, is
$$\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} + \frac{1}{\sqrt{\gamma}} = 0,$$
that its tangential equation is
$$(u + v + w)^3 = 27uvw,$$
and that its reciprocal polar with regard to its centre is
$$r \sin 3\theta = c.$$

Curves rolling on fixed curves.

28. *To find the path of any given point in the area of a plane curve which rolls on a fixed curve.*

If $O'P$ be the rolling curve, O' having been coincident with O, fig. (18), let x, y, be coordinates of P referred to the normal and tangent at O, and x', y', referred similarly to O'; ϕ, ϕ', the angles which the normal at P makes with the normals at O, O'.

Then, if α, β are the coordinates of O', referred to O,

$$\alpha = y' \sin(\phi + \phi') - x' \cos(\phi + \phi') - x.$$

$$\beta = y - y' \cos(\phi + \phi') - x' \sin(\phi + \phi'),$$

and, as the right-hand members of these equations can be found in terms of the arc OP (s), the relation between α and β can be found.

If Q be a point, the coordinates of which, referred to O', are (a, b), and if ξ, η, be the coordinates of Q referred to O,

$$\xi = \alpha - a \cos(\phi + \phi') - b \sin(\phi + \phi')$$

$$\eta = \beta + a \sin(\phi + \phi') + b \cos(\phi + \phi').$$

29. If $QP = r$, and $PQO' = \theta$, the relation between r and θ is the polar equation of the rolling curve referred to Q and QO'.

Taking ξ and η as the coordinates of Q referred to O, and ψ as the angle QEN,

$$\xi = r \cos\psi - x, \text{ and } \eta = r \sin\psi + y.$$

Now $$\tan\psi = -\frac{d\xi}{d\eta}, \text{ and } \tan PTO = \frac{dy}{dx},$$

$$\therefore r\frac{d\theta}{dr} = \tan QPT = \tan(PET + PTE)$$

$$= \frac{\dfrac{dy}{dx} + \dfrac{d\xi}{d\eta}}{1 + \dfrac{dy}{dx} \cdot \dfrac{d\xi}{d\eta}}.$$

If the fixed and rolling curves are given, $\frac{d\theta}{dr}$ can be formed in terms of r, and $\frac{dy}{dx}$ in terms of x, and then the elimination of x and r will give the differential equation of the locus of Q.

If the locus of Q and the fixed curve are given, the elimination of x and ξ will give the rolling curve.

If the locus of Q and the rolling curve are given, the elimination of x and ξ_1 will give the fixed curve.

30. *A curve rolls on a straight line; it is required to find the curvature of the path of any point carried with it.*

Let QP, $Q'P'$ be consecutive normals to the path of Q (fig. 9).

Their intersection E is ultimately the centre of curvature.

Let $Q'P'$ be the changed position of Qp.

Then $Q'LQ =$ the angle through which the curve has turned

$=$ the angle through which the normal at P has turned

$= \frac{\delta s}{\rho}$, if ρ be the radius of curvature at P;

$$\therefore EQ = \frac{\text{arc } QQ'}{\angle E} = \frac{QQ'}{QLQ' - LQP}$$

$$= \frac{QL \cdot \frac{\delta s}{\rho}}{\frac{\delta s}{\rho} - \frac{\delta s \cdot \cos\alpha}{r}},$$

if α be the angle between QP (r) and the normal at P,

$$= \frac{r^2}{r - \rho\cos\alpha},$$

ultimately, observing that since the displacement of p from P' is of the second order, we may in this case assume that p and P' are coincident; Arts. (3) and (4).

Thus, in the case in which the roulette is a cycloid,

$$r = 2a\cos\alpha, \text{ and } \rho = a;$$

$$\therefore EQ = 2r = 2PQ.$$

31. *The focus-roulettes of a conic section on a straight line.*

Let Q, fig. (9), be the focus of a conic rolling on a straight line.

Then $\rho \cos \alpha$ is the chord of curvature through the focus.

If the conic is an ellipse,

$$\rho \cos \alpha = \frac{CD^2}{AC} = \frac{r(2a-r)}{a},$$

and therefore, if ρ' be the radius of curvature of the focus-roulette,

$$\frac{1}{\rho'} + \frac{1}{r} = \frac{1}{a}.$$

If the conic is an hyperbola,

$$\rho \cos \alpha = \frac{r(2a+r)}{a}$$

and therefore $$\frac{1}{\rho'} + \frac{1}{r} = -\frac{1}{a},$$

shewing that the roulette is convex to the straight line.

If the conic is a parabola,

$$\rho \cos \alpha = 2r,$$

and therefore $$\rho' = -r,$$

the known property of a catenary, the directrix of which is the straight line.

We can however deduce the equation of the catenary.

For, if ϕ is the inclination of the tangent at Q to the fixed line, $\phi = \alpha$,

and $$\frac{2a}{r} = 1 - \cos 2\left(\frac{\pi}{2} - \phi\right) = 2\cos^2 \phi.$$

Hence $$\frac{ds}{d\phi} = r = a \sec^2 \phi,$$

and therefore $s = a \tan \phi$.

32. The preceding formula may also be obtained by help of the equation

$$\frac{ds}{d\phi} = p + \frac{d^2 p}{d\phi^2}.$$

For (fig. 19), if $OPQ=\phi$, and $OP=AP=s$,

$$p=r-s\cos\phi,$$

and
$$\frac{dp}{d\phi}=OZ=s\sin\phi;$$

$$\therefore\ p+\frac{d^2p}{d\phi^2}=r+\sin\phi\frac{ds}{d\phi}.$$

But if $AQP=\theta$, and $AFP=\psi$,

$$\theta=\psi+\frac{\pi}{2}-\phi;$$

therefore radius of curvature of roulette

$$=r+\cos\alpha\frac{ds}{d\psi-d\theta};$$

and since $r\dfrac{d\theta}{ds}=\sin\phi$, and $\dfrac{ds}{d\psi}=\rho$,

this
$$=r+\frac{\cos\alpha}{\dfrac{1}{\rho}-\dfrac{1}{r}\cos\alpha}=\frac{r^2}{r-\rho\cos\alpha}.$$

We may observe that the curvature of the roulette is zero if $r=\rho\cos\alpha$, that is, if the point is situated on the circle of which ρ is the diameter.

33. The following theorem is of great importance.

If a curve roll on a fixed curve over a small arc δs, *the angle turned through by any line in the plane of the rolling curve is* $\delta s\left(\dfrac{1}{\rho}+\dfrac{1}{\rho'}\right)$, *where* ρ *and* ρ' *are the radii of curvature of the fixed and rolling curves.*

Let $Pp=PP'=\delta s$, fig. (20), and let the normals OP, $O'P'$, meet in L.

Then, since OP turns into the position KLP', the angle OLK between the lines is the angle required, and this angle

$$=POp+P'O'P$$

$$=\frac{\delta s}{\rho}+\frac{\delta s}{\rho'}.$$

If the concavities are in the same direction, and if the rolling curve is inside the fixed curve the angle turned through is

$$\delta s\left(\frac{1}{\rho'}-\frac{1}{\rho}\right),$$

but, if the rolling curve is outside the fixed curve, the angle turned through is

$$\delta s\left(\frac{1}{\rho}-\frac{1}{\rho'}\right).$$

34. *A curve rolls on a fixed curve; to find the curvature of the roulette traced by any point carried with it.*

Q being the carried point, the angle QLQ' = the angle of displacement $= \delta s\left(\frac{1}{\rho}+\frac{1}{\rho'}\right)$, (fig. 21).

$$EQ=\frac{QQ'}{QEQ'}=\frac{QL\,(QLQ')}{QLQ'-PQL},$$

and, since the displacement of p is of the second order,

$$PQL=\frac{PP'\cos\alpha}{PQ}=\frac{\delta s\cos\alpha}{r},$$

taking α as the inclination of QP to the normal at P.

$$\therefore R=EQ=\frac{r\left(\frac{1}{\rho}+\frac{1}{\rho'}\right)}{\frac{1}{\rho}+\frac{1}{\rho'}-\frac{\cos\alpha}{r}}.$$

If the curve roll inside the fixed curve the expression will be

$$\frac{r\left(\frac{1}{\rho}-\frac{1}{\rho'}\right)}{\frac{1}{\rho}-\frac{1}{\rho'}-\frac{\cos\alpha}{r}}.$$

We observe that the curvature of the roulette vanishes if

$$r=\rho\rho'\cos\alpha/(\rho+\rho'),$$

and therefore there is a point of inflection of the roulette if the point P is situated on the circle of which $\rho\rho'/(\rho+\rho')$ is the diameter.

35. Referring to the same figure, let ψ be the inclination of QP to the fixed normal OF, A the point which has passed over O, $AQP=\theta$, $QP=r$.

Then, if $r=f(\theta)$ is known, and if $s=F(\phi)$ is the equation of OP, EQ can be found in terms of s and therefore of ϕ.

Also $$\psi=\phi-\alpha;$$

$\therefore \dfrac{d\sigma}{d\psi}$ or EQ can be found in terms of ψ, and this gives the intrinsic equation of the path of Q.

36. Thus, for an epicycloid,

radius of curvature of path of Q (fig. 12),

$$=PQ\frac{\dfrac{1}{a}+\dfrac{1}{b}}{\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{\cos\alpha}{PQ}},$$

or $$R=PQ\,.\frac{\dfrac{1}{a}+\dfrac{1}{b}}{\dfrac{1}{a}+\dfrac{1}{2b}}=2PQ\,.\frac{a+b}{a+2b}.$$

Again, take the case of an ellipse rolling on an equal ellipse, corresponding points being in contact, and consider the path of a focus S.

If P be the point of contact,

$$\cos\alpha=\frac{PF}{AC},\quad \rho=\rho'=\frac{CD^2}{PF};$$

$$\therefore\ R=\frac{2\dfrac{PF}{CD^2}\,.\,SP}{\dfrac{2PF}{CD^2}-\dfrac{PF}{AC\,.\,SP}}=2AC,$$

as is *à priori* obvious.

37. *If a curve roll on a straight line, the arc of the roulette is equal to the corresponding arc of the pedal.*

The angle turned through (fig. 22),

$$= \text{angle between normals at } P \text{ and } p,$$
$$= YQY';$$
$$\therefore \text{ arc } QQ' = QP \,.\, YQY'$$
$$= QT \,.\, YQY'$$
$$= \frac{YY'}{\sin YQY'} \,.\, YQY' = YY';$$

QT being the diameter of the circle about YQY'.

Or thus, if $QY = y$,

$$\frac{dy}{ds} = \cos QPY,$$

and if $\sigma =$ arc of pedal, y being the radius vector,

$$\frac{dy}{d\sigma} = \cos QYZ = \cos QPY;$$
$$\therefore \; ds = d\sigma.$$

38. *If a curve roll on a fixed curve, the element of arc of the roulette is to the corresponding arc of the pedal as $\rho + \rho' : \rho'$, ρ being the radius of curvature of the rolling curve, and ρ' of the fixed curve.*

Imagining the line OT to be the fixed curve (fig. 22), the angle turned through $= ds\left(\frac{1}{\rho} + \frac{1}{\rho'}\right)$;

$$\therefore \; QQ' = QP \,.\, ds\left(\frac{1}{\rho} + \frac{1}{\rho'}\right).$$

Also, QT being the diameter of the circle YQY',

$$YY' = QT \,.\, \sin YQY' = QP \,.\, \frac{ds}{\rho};$$

$$\therefore \; QQ' : YY' = \frac{1}{\rho} + \frac{1}{\rho'} : \frac{1}{\rho} = \rho + \rho' : \rho'.$$

Hence the length of the arc of the roulette

$$=\int\left(1+\frac{\rho}{\rho'}\right)d\sigma,$$

where $d\sigma$ is an element of the arc of the pedal.

In the case of cycloidal, or trochoidal curves, ρ and ρ' are constant, and the arc of the roulette is proportional to the arc of the pedal.

In the case of a curve rolling on an equal curve, corresponding points being in contact, the arc of the roulette is always double that of the pedal.

Also the roulette of any point is similar to, and double of the pedal.

39. *If a curve roll on a straight line, the area between the roulette, the fixed line, and any two ordinates, is double the corresponding pedal area.*

For, if $Y'N$ be the perpendicular from Y' on PY (fig. 22), the element of area $= QY \,.\, YN$, neglecting infinitesimals of the second order,

$$\begin{aligned} &= QY \,.\, YY' . \cos TYY' \\ &= QY \,.\, YY' . \sin QYY' \\ &= 2\Delta QYY'. \end{aligned}$$

Or thus, if x, y be co-ordinates of Q,

$$\frac{ydx}{ds} = y \sin QPY = y \sin QYZ = QZ = p\,;$$

$$\begin{aligned} \therefore\ ydx &= pds = pd\sigma \\ &= 2\ (\text{element of polar area}). \end{aligned}$$

40. *To find the area swept over by the normal* QP.

Taking figure (22), let $QP = r$, $PQp = \delta\theta$, and $\delta\phi =$ the angle of deflection from P to p, which is equal to the angle between Qp and $Q'P'$.

Then the area $QPP'Q' = QPP' + QP'Q'$

$$= QPp + QpQ',$$

observing that pP' is of the second order,

$$= \frac{1}{2} r^2 \delta\theta + \frac{1}{2} r^2 \delta\phi;$$

therefore the area between the roulette, the fixed line and two normals

$$= \frac{1}{2}\int r^2 d\theta + \frac{1}{2}\int r^2 d\phi.$$

If the curve be a closed curve and make one revolution, this area = area of curve $+ \frac{1}{2}\int_0^{2\pi} r^2 d\phi$.

Hence, if the rolling commence when QP is perpendicular to the fixed line,

$$2\,(\text{area of pedal}) = \text{area of curve} + \frac{1}{2}\int_0^{2\pi} r^2 d\phi;$$

$$\text{or area of curve} = \int_0^{2\pi} \left(p^2 - \frac{1}{2} r^2\right) d\phi.$$

Take for example the case of a cycloid.

The area swept over by QP, fig. (4),

$$= \frac{1}{2}\int r^2 d\theta + \frac{1}{2}\int r^2 d\phi = \frac{3}{4}\int r^2 d\theta$$

$$= 3a^2 \int \cos^2\frac{\theta}{2}\, d\theta = \frac{3a^2}{2}(\theta + \sin\theta),$$

and, if $\theta = \pi$, this $\qquad = \frac{3}{2}\pi a^2,$

so that the whole area of the cycloid is $3\pi a^2$.

41. *A curve rolls on a fixed curve; to find the area swept over by the normal* QP.

If the arc $Pp = PP'$ (fig. 23), then, pP' being of the second order,

$$\text{area } QPP'Q' = QPp + QpQ'$$
$$= \frac{1}{2} r^2 \delta\theta + \frac{1}{2} r^2 \left(\frac{\delta s}{\rho} + \frac{\delta s}{\rho'}\right);$$

therefore area swept over by QP

$$= \int \frac{1}{2} r^2 d\theta + \int \frac{1}{2} r^2 ds \left(\frac{1}{\rho} + \frac{1}{\rho'}\right).$$

Hence, if the curve be a closed oval, and if it make a complete revolution, the area between the arc of the roulette, the two normals at its ends, and the curve

$$= \text{area of curve} + \frac{1}{2} \int r^2 \delta s \left(\frac{1}{\rho} + \frac{1}{\rho'}\right).$$

42. *To find the locus of the centre of curvature at the point of contact of a curve rolling on a straight line.*

Let x, y be co-ordinates of the centre of curvature, then, if $s = f(\phi)$ be the rolling curve,

$$x = s = f(\phi),$$

and $$y = \rho = f'(\phi),$$

whence, eliminating ϕ, the locus is obtained.

Thus, if the curve be an epicycloid, or hypocycloid, the locus is an ellipse.

If it be a catenary, the locus is a parabola, and if it be an equiangular spiral, the locus is a straight line.

43. If the curve roll on a fixed curve, $s = F(\phi')$, and if $s = f(\phi)$ be the rolling curve,

$$x = OM + \rho \sin \phi' \text{ (fig. 24)},$$
$$y = \rho \cos \phi' - PM;$$

therefore, if OM and PM can be found in terms of ϕ', we have, with $f(\phi) = F(\phi')$, three equations from which ϕ and ϕ' may be eliminated.

Suppose, for example, the curves to be equal catenaries, their vertices at first coinciding.

Then, $\rho = c\sec^2\phi$, and $PM = c\sec\phi - c$;

$$\therefore\ y = c.$$

The locus is therefore a straight line, as is *à priori* obvious, if we remember that the normal bounded by the directrix is equal to the radius of curvature.

If the curves be equal cycloids, their vertices at first coinciding,

$$OP = 4a\sin\phi, \quad PE = 4a\cos\phi,$$

$$OM = a(2\phi + \sin 2\phi), \text{ and } PM = a(1 - \cos 2\phi).$$

$$\therefore\ ON = 2a\phi + 3a\sin 2\phi, \text{ and } NE = a + 3a\cos 2\phi,$$

so that the locus of E is the same as the locus of a point carried by a circle of radius a rolling on a straight line, the point being at the distance $3a$ from the centre of the circle.

44. To find the length of the curve formed by the successive positions of the centre of curvature we may proceed as follows.

Let Q be the centre of curvature at P, q at p, and Q' the position of q when the curve has rolled from P to P', so that QQ' is an element of the locus (fig. 25).

Then q may be taken to be in the normal PQ, since its distance from PQ, the tangent to the evolute at q, is an infinitesimal of the second order.

Hence, if $PQ = \rho$, and if ρ' be the radius of curvature of the fixed curve at P,

$$Qq = \delta\rho, \quad qQ' = (\rho + \rho')\frac{ds}{\rho'},$$

and the inclination to PQ of the tangent at Q

$$= \tan^{-1}\frac{(\rho + \rho')\,ds}{\rho' d\rho}.$$

Also, if
$$QQ' = \delta\sigma,$$
$$\delta\sigma^2 = \delta\rho^2 + \left(\frac{\rho + \rho'}{\rho'}\right)^2 \delta s^2.$$

Hence we can find the intrinsic equation, for if ψ be the inclination of the tangent QQ' to the tangent at a fixed point O,
$$\psi = \phi - \frac{\pi}{2} + \tan^{-1}\frac{(\rho + \rho')\,ds}{\rho' d\rho},$$
ϕ being the deflection of the fixed curve from O to P.

As an example, again take the case of the equal catenaries; then
$$\rho = \rho' = c\sec^2\phi, \quad \delta s = \rho\delta\phi,$$
and
$$\psi = \phi - \frac{\pi}{2} + \tan^{-1}(\cot\phi) = 0,$$
so that QQ' is parallel to the tangent at O, as already seen.

45. *Envelope Roulettes.*

We have hitherto considered only the roulettes produced by points carried with a rolling curve; we now proceed to consider the roulettes enveloped by lines carried with a rolling curve.

A curve rolls on a straight line, to find the envelope of any straight line carried with it.

If P be the point of contact, and PQ the perpendicular let fall from P on the carried line, Q is the point of contact of its envelope (fig. 26).

Let pq be the perpendicular from a consecutive point p, then as the curve rolls over PP', q is carried to Q', and if $\delta\sigma$ be an arc of the roulette enveloped,
$$\begin{aligned}\delta\sigma &= Qq + qQ' \\ &= \sin\phi\delta s + r\delta\phi,\end{aligned}$$
if $OPQ = \phi$, and $PQ = r$.

Hence the radius of curvature $= \dfrac{d\sigma}{d\phi}$

$$= r + \sin\phi \frac{ds}{d\phi}$$

$$= r + \rho \sin\phi.$$

For example, consider the roulette produced by a diameter of a circle rolling on a straight line.

Then $$r = a\sin\phi, \quad \rho = a,$$

$$\frac{d\sigma}{d\phi} = 2a\sin\phi,$$

and the roulette is a cycloid.

Ex. 2. *A parabola rolls on a straight line, it is required to find the envelope of the latus rectum.*

In this case, $\rho\sin\phi = 2SP$ (fig. 27), and

$$PQ = a - x;$$

$$\therefore \frac{d\sigma}{d\phi} = a - x + 2(a + x)$$

$$= 3a + a\cot^2\phi,$$

$$\therefore = 2a + a\operatorname{cosec}^2\phi;$$

$$\therefore \sigma = 2a\phi - a\cot\phi + C,$$

and the length of the roulette between the two points at which it cuts the fixed line, i.e. from

$$\phi = \frac{\pi}{4} \text{ to } \phi = \frac{3\pi}{4}, \text{ is } (\pi + 2)a.$$

46. The tangential polar equation may be obtained, directly.

Thus, if $s = OP = \text{arc } AP$, and $p = OY$ the perpendicular on the carried line,

$$p = r - s\cos\phi,$$

whence the equation, if r and s be known in terms of ϕ.

Or, if s only be known, and OZ be the perpendicular on PQ, fig. 26,

$$OZ = \frac{dp}{d\phi},$$

and therefore $$\frac{dp}{d\phi} = s \sin \phi.$$

Hence the radius of curvature

$$\begin{aligned} &= p + \frac{d^2 p}{d\phi^2} \\ &= r - s \cos \phi + s \cos \phi + \sin \phi \frac{ds}{d\phi} \\ &= r + \rho \sin \phi, \end{aligned}$$

as before.

47. *A curve rolls on a given curve, carrying a straight line; to find the roulette enveloped.*

Making the same construction as before, and observing that the displacement of p is of the second order (fig. 28),

$$\begin{aligned} \delta\sigma &= Qq + qQ' \\ &= \delta s \cos \alpha + r\delta\phi, \end{aligned}$$

where $\delta\phi = \delta s \left(\frac{1}{\rho} + \frac{1}{\rho'}\right)$, $PQ = r$ and α is the inclination of PQ to the normal at P.

Hence the radius of curvature of the roulette at Q

$$= \frac{d\sigma}{d\phi} = r + \cos \alpha \frac{\rho\rho'}{\rho + \rho'},$$

and if r, α, ρ and ρ' can be found in terms of the angle which PQ makes with some fixed line, the intrinsic equation can be found.

If the concavities are in the same direction, the expression for the radius of curvature of the envelope roulette is

$$r + \cos \alpha \frac{\rho\rho'}{\rho' - \rho}.$$

48. Example. *A circle rolls outside a fixed circle; to find the length of the curve enveloped by a diameter.*

If AD be the diameter, A passing over A', and if $A'OP = \theta$ (fig. 29),

$$PQ = b \sin \frac{a\theta}{b},$$

and
$$\frac{d\sigma}{d\phi} = b \sin \frac{a\theta}{b} + \cos CPQ \frac{ab}{a+b}$$

$$= \frac{b^2 + 2ab}{a+b} \sin \frac{a\theta}{b}.$$

The angle of deflection of AD from OA'

$$= \phi = \theta + \frac{a\theta}{b};$$

$$\therefore \frac{d\sigma}{d\phi} = \frac{b^2 + 2ab}{a+b} \sin \frac{a\phi}{a+b} \dots\dots\dots\dots\dots (\alpha),$$

and
$$\sigma = \frac{b^2 + 2ab}{a}\left(1 - \cos \frac{a\phi}{a+b}\right),$$

measuring from A'.

Taking a half roll of the circle, that is from $\theta = 0$ to $\theta = \frac{\pi b}{a}$, we get the length of the arc from one cusp to the next, which is therefore

$$2\frac{b}{a}(b + 2a).$$

Comparing the equation (α) with that of Art. (21), we observe that the envelope of AD is the epicycloid which would be produced by a circle of radius $\frac{b}{2}$ rolling on the circle O.

This can also be seen geometrically, for, describing a circle on PC as diameter, the arc PQ is equal to the arc PA, and therefore to the arc PA'.

49. *A curve rolls on an equal curve, corresponding points coinciding; to find the envelope of any normal of the rolling curve.*

Let $\phi = BTQ$ (fig. 30), and measure ψ from the tangent at O, the point corresponding to A.

Then, for the envelope of the normal at A,

$$\frac{d\sigma}{d\phi} = r + \cos\alpha\frac{\rho}{2}, \quad \text{Art. (47)},$$

and, if $s = f(\psi)$ be the curve,

$$r = \int_0^\psi \cos\psi ds, \ \rho = f'(\psi), \text{ and } \phi = 2\psi;$$

$$\therefore \frac{d\sigma}{d\phi} = \int_0^{\frac{\phi}{2}} \cos\psi f'(\psi) d\psi + \frac{1}{2}\sin\frac{\phi}{2} f'\left(\frac{\phi}{2}\right).$$

For example, let a circle roll on an equal circle; then

$$s = a\frac{\phi}{2},$$

$$\frac{d\sigma}{d\phi} = \int_0^{\frac{\phi}{2}} a\cos\theta d\theta + \frac{a}{2}\sin\frac{\phi}{2} = \frac{3a}{2}\sin\frac{\phi}{2},$$

and $$\sigma = 3a\left(1 - \cos\frac{\phi}{2}\right),$$

a two-cusped epicycloid.

Taking a half roll of the circle the arc is $6a$, which agrees with the result in Art. (48), putting $b = a$.

50. *A curve rolls on a fixed curve; it is required to find the envelope of any curve carried with it.*

If P be the point of contact, draw from P normals to the carried curve (fig. 31).

Then, if PQ be one of these normals, it is the normal at Q to the envelope, and the other normals similarly belong to other portions of the envelope.

Let pq, the normal from a consecutive point p, roll into the position $P'Q'$; then E, the intersection of PQ and $P'Q'$, is the centre of curvature.

The arc $QQ' = \delta\sigma = Qq + r\delta\phi$, where $r = PQ$ and

$$\delta\phi = \delta s\left(\frac{1}{\rho} + \frac{1}{\rho'}\right).$$

Let ρ'' be the radius of curvature at Q of the carried curve; then

$$Qq : \delta s . \cos\alpha :: \rho'' : r + \rho'',$$

α being the inclination of QP to the normal at P;

and the angle $PEP' = \delta\phi - \dfrac{Qq}{\rho''}$

$$= \delta s\left(\frac{1}{\rho} + \frac{1}{\rho'}\right) - \delta s\,\frac{\cos\alpha}{r + \rho''};$$

therefore EQ, the radius of curvature of the envelope,

$$= \frac{\dfrac{\rho''\cos\alpha}{r + \rho''} + r\left(\dfrac{1}{\rho} + \dfrac{1}{\rho'}\right)}{\dfrac{1}{\rho} + \dfrac{1}{\rho'} - \dfrac{\cos\alpha}{r + \rho''}}.$$

Making ρ'' infinite, this of course gives the formula of Art. (47), and if ρ'' vanish, the formula of Art. (34) results.

If the various quantities involved in this expression can be found in terms of ψ, the angle of deflection of PQ, the intrinsic equation is determined.

If any of the curves instead of being convex, as in the figure, be concave, the signs of ρ, &c. must be changed.

51. Ex. 1. *A curve rolls on another, carrying a parallel curve.*

In this case, $\alpha = 0$, $r = d$, and $\rho'' = \rho - d$.

Hence EQ becomes $d + \rho'$, a result which is obviously true.

Ex. 2. *A circle, of radius* c, *rolls inside an oval curve.*

If $\delta\sigma$ be an elementary arc of the envelope, and δs the arc of the curve rolled over,

$$\delta\sigma = 2c\delta\phi - \delta s,$$

where
$$\delta\phi = \delta s\left(\frac{1}{c} - \frac{1}{\rho}\right),$$

so that
$$\delta\sigma = \delta s - 2c\,\frac{\delta s}{\rho},$$

and therefore total length of envelope $= p - 4\pi c$, where p is the perimeter of the oval curve.

Ex. 3. *A straight line rolls on a fixed circle, carrying an equal circle with which it is in contact.*

Let $A'OP = \theta$ (fig. 32), then

$$r = CP - a = a\sqrt{1+\theta^2} - a,$$

$$\cos\alpha = \frac{a}{CP} = \frac{1}{\sqrt{1+\theta^2}},$$

$$\rho = \infty, \quad \rho' = \rho'' = a,$$

and therefore
$$\frac{d\sigma}{d\psi} = a\,\frac{\dfrac{1}{1+\theta^2} + \sqrt{1+\theta^2} - 1}{1 - \dfrac{1}{1+\theta^2}}$$

$$= a\left(\frac{\overline{1+\theta^2}|^{\frac{3}{2}} - \theta^2}{\theta^2}\right).$$

Also, if ψ be the inclination of PQ to OA',

$$\psi = \theta - \alpha = \theta - \tan^{-1}\theta;$$

$$\therefore\ d\psi = d\theta - \frac{d\theta}{1+\theta^2},$$

and
$$\frac{d\sigma}{d\theta} = a\left(\frac{\overline{1+\theta^2}|^{\frac{3}{2}}}{\theta^2} - 1\right)\frac{\theta^2}{1+\theta^2} = a\left(\sqrt{1+\theta^2} - \frac{\theta^2}{1+\theta^2}\right).$$

This equation, when integrated, determines the length of the envelope.

If we put $-a$ for ρ'', and $CP+a$ for r, we shall obtain the other portion of the envelope due to the normal PQ'.

52. *A curve rolls on a straight line; to find the area between the straight line, the envelope of any carried straight line, and two normals of the envelope.*

The element of area $PQQ'P'$ (fig. 26),

$$= PQqp + P'qQ'$$

$$= r\delta s \sin\phi + \frac{1}{2}r^2\delta\phi$$

$$= \left(r\rho\sin\phi + \frac{1}{2}r^2\right)\delta\phi,$$

the integration of which expression gives the area if the intrinsic equation of the rolling curve be known.

If the line PQ fall below the line OP, the element of area swept over will be

$$\left(r\rho\sin\phi - \frac{1}{2}r^2\right)\delta\phi.$$

This however is included in the former, if we suppose r to be an algebraic expression for PQ.

53. *A curve rolls on another; to find the area between the envelope of any carried curve, any two normals of the envelope, and the fixed curve.*

The element of area $= PQQ'P'$ (fig. 31),

$$= PQqp + \frac{1}{2}r^2\delta\phi,$$

where
$$\delta\phi = \delta s\left(\frac{1}{\rho} + \frac{1}{\rho'}\right).$$

Hence the area swept over exceeds the area between the curve and the carried curve by $\int \frac{1}{2}r^2 d\phi$.

Thus in the case of Ex. 3 of Art. 51,

$$r^2 = a^2 (2 + \theta^2 - 2\sqrt{1+\theta^2}),$$

and
$$d\psi = d\theta - \frac{d\theta}{1+\theta^2};$$

therefore the area of the envelope exceeds the area APQ by

$$\frac{a^2}{2}\int (2 + \theta^2 - 2\sqrt{1+\theta^2})\, d\psi.$$

54. Adopting the notation of preceding articles, we can give an expression for the element $PQqp$ (fig. 31).

For this element

$$= \left\{\frac{1}{2}(r+\rho'')^2 - \frac{1}{2}\rho''^2\right\}\frac{\delta s''}{\rho''}$$

$$= \left(r\rho'' + \frac{r^2}{2}\right)\frac{\delta s''}{\rho''},$$

and
$$\delta s'' : \delta s \,.\, \cos\alpha :: \rho'' : r + \rho'';$$

therefore element

$$PQqp = \left(\rho'' + \frac{r}{2}\right)\frac{r\cos\alpha\delta s}{r+\rho''}.$$

55. The following examples will serve as additional illustrations of the preceding methods.

A cycloid rolls on a straight line; it is proposed to consider the roulette enveloped by the tangent at its vertex.

The cycloid is $s = 4a\sin\phi$, and for the envelope of AQ (fig. 33),

$$\frac{d\sigma}{d\phi} = \rho\cos\phi - PQ$$

$$= 4a\cos^2\phi - 2a\sin^2\phi$$

$$= a + 3a\cos 2\phi;$$

$$\therefore\ \sigma = a\phi + \frac{3a}{2}\sin 2\phi,$$

measuring from O.

To trace the curve, observe that there is a cusp when

$$\cos 2\phi = -\frac{1}{3},$$

i.e., when ϕ is a little greater than $\frac{\pi}{3}$, and that the curve cuts the initial line at distances $2a$ from O: also that for one roll of the cycloid the curve lies wholly below the initial line.

If the cycloid be continued, and the rolling go on continuously, the next branch of the cycloid will give the next half of the roulette above the initial line, after which the successive branches of the cycloid will produce continually the same roulette.

Further, the evolute of the roulette is

$$s = 3a \cos 2\phi,$$

a four-cusped hypocycloid.

The curve may be further examined by finding x and y, the co-ordinates of Q referred to O, viz.

$$x = 4a \sin\phi - 2a \sin^3\phi, \qquad y = a \sin\phi \sin 2\phi.$$

Fig. 34 represents the roulette, and its evolute.

The element of area swept over by PQ

$$= PQ\delta s \cos\phi - \frac{1}{2} PQ^2 \delta\phi$$

$$= 4a^2 \sin^2\phi\, (2 - 3\sin^2\phi)\, \delta\phi,$$

which becomes negative when $\sin^2\phi > \frac{2}{3}$, or when

$$\cos 2\phi = -\frac{1}{3},$$

i.e. at the cusp.

The integration from $\phi = 0$ to $\sin^2\phi = \frac{2}{3}$ gives the area OET, and from $\sin^2\phi = \frac{2}{3}$ to $\phi = \frac{\pi}{2}$ the difference between the areas AET and CET.

If we wish to find the area enclosed by the roulette, it will be at once given by integrating the expression,

$$\frac{1}{2}(xdy - ydx), \text{ from } \phi = 0 \text{ to } \phi = \frac{\pi}{2}.$$

This expression

$$= a^2 (6 \sin^4 \phi - 4 \sin^2 \phi)\, d\phi,$$

the integral of which between the assigned limits

$$= a^2 \left(6 \cdot \frac{3}{4} \cdot \frac{1}{2} - 4 \cdot \frac{1}{2}\right) \frac{\pi}{2} = \frac{\pi}{8} a^2,$$

and the complete area inclosed by the roulette $= \frac{1}{2} \pi a^2$, i.e. it is half the area of the generating circle of the cycloid.

To find the length of the arc of the roulette, we must take ϕ from O to $\sin^{-1}\sqrt{\frac{2}{3}}$, and then from $\phi = \sin^{-1}\sqrt{\frac{2}{3}}$ to $\phi = \frac{\pi}{2}$, and add together the numerical values of the results. This will give one-fourth of the whole arc.

If ϕ be taken from O to $\frac{\pi}{2}$ we obtain $\sigma = \frac{\pi a}{2}$, so that the difference between the arcs OE and EC of the roulette is one quarter of the perimeter of the generating circle of the cycloid.

56. *A circle rolls on an equal circle, carrying a tangent; it is required to determine the nature of the roulette produced by the tangent.*

Let $OY = p$ (fig. 35), $A'OP = ACP = \theta$, and $YOA' = \phi$, OY being the perpendicular on the carried line.

Then $$p = 2a \cos\theta - a, \text{ and } \phi = 2\theta;$$

$$\therefore p = 2a \cos\frac{\phi}{2} - a,$$

the tangential polar equation of the envelope.

Next $$\frac{d\sigma}{d\phi} = p + \frac{d^2p}{d\phi^2} = \frac{3a}{2}\cos\frac{\phi}{2} - a;$$

$$\therefore \sigma = 3a \sin\frac{\phi}{2} - a\phi,$$

the intrinsic equation, measuring σ from A'

Observing that the radius of curvature

$$= \frac{3a}{2}\cos\theta - a,$$

we see that a cusp occurs when $\cos\theta = \frac{2}{3}$. There are therefore two cusps, corresponding to the positive and negative values of θ.

When $\theta = \frac{\pi}{3}$, $p = 0$.

When $\cos\theta = \frac{1}{3}$, it will be seen by a figure that the tangent passes through the other end of the diameter through P, and that the envelope then crosses the circle at a point V distant $\frac{2a}{3}$ from P. It will also be found that the tangent at the cusp meets the diameter $A'O$ in the same point T at which it is intersected by the envelope itself.

Putting together all these considerations we obtain the figure (fig. 36), the curve being an involute of a two-cusped epicycloid.

The element of area swept over by PQ, i.e.

$$\begin{aligned} PQQ'P' &= r\cos\alpha\delta s - \frac{1}{2}r^2\delta\phi \\ &= r\delta\theta\,(a\cos\alpha - r) \\ &= r\delta\theta\,(2a\cos\theta - a) \\ &= pr\delta\theta. \end{aligned}$$

When $\theta > \frac{\pi}{3}$, this expression is still the same if we write for p its numerical value $a - 2a\cos\theta$, and any portion of the area is thus found by a simple integration.

EXAMPLES.

1. If a cycloid roll on the tangent at the vertex, the locus of the centre of curvature at the point of contact is a semicircle whose radius is four times that of the generating circle.

2. Prove that a cardioid is an epicycloid due to the rolling of a circle, with internal contact on a fixed circle of half its diameter.

3. The roulette, on a straight line, of the pole of an epicycloid is an ellipse.

4. The roulette, on a circle, of the pole of an equiangular spiral, is an involute of another circle.

5. When a curve rolls on a straight line, shew how to find the locus of the centre of curvature at the point of contact, and prove that, in the case of a cardioid, the locus is an ellipse.

When a curve rolls on a fixed curve, prove that the locus of the centre of curvature of the rolling curve at the point of contact is inclined to the common tangent at the angle $\tan^{-1}\{\rho' d\rho/(\rho+\rho')\,ds\}$, where ρ, ρ' are the radii of curvature of the fixed and rolling curves at the point of contact.

6. A curve A rolls on a curve B so that its pole describes a straight base, and the curvatures of A and B at the point of contact are always as n to 1, estimated in the same direction. Prove that the radius of curvature of B is $n-1$ times the normal terminated by the base, and that the chord of curvature of A through the pole is to the radius vector in the ratio of $2(n-1)$ to n.

Prove also that if A and B roll on a straight line, the roulette of the pole of A is the envelope of the base carried by B, and that the radius of curvature of the roulette is n times the normal terminated by the same straight line.

State what curves A and B are when n is, -1, 0, 1, 2, 3.

7. A cycloid rolls on an equal cycloid, corresponding points being in contact; prove that the locus of the centre of curvature of the rolling curve at the point of contact is a trochoid whose generating circle is equal to that of either cycloid.

8. A circle rolls on a straight line; prove that the envelope of any carried straight line is an involute of a cycloid; and trace the figures corresponding to the cases in which the distance of the carried line from the centre is greater than, equal to, or less than the diameter of the circle.

9. A straight line rolls on the curve, $s = f(\phi)$, carrying a straight line inclined to it at the angle α; the envelope roulette is

$$s = \sin \alpha f(\phi) + \cos \alpha f'(\phi).$$

If the curve be an epicycloid, or hypocycloid, the envelope roulette is of the same class.

10. The roulette, on a straight line, of the pole of the hyperbolic spiral, $r\theta = c$, is

$$\frac{dy}{dx} = \frac{y}{\sqrt{c^2 - y^2}};$$

and of the pole of the curve, $c^n p = r^{n+1}$, is

$$\left(\frac{ds}{dx}\right)^{n+1} = \left(\frac{c}{y}\right)^n.$$

11. The roulette, on a straight line, of the pole of a cardioid is

$$4a - x = \{2(2a)^{\frac{2}{3}} + y^{\frac{2}{3}}\}\{(2a)^{\frac{2}{3}} - y^{\frac{2}{3}}\}^{\frac{1}{2}}.$$

and its area is $\dfrac{15}{2}\pi a^2$.

12. A parabola rolls symmetrically on an equal parabola; find the path of the focus, and prove that the path of the vertex is the cissoid

$$y^2(2a - x) = x^3.$$

13. An involute of a circle rolls on a straight line; the roulette of the centre of the circle is a parabola.

14. The ellipse $\frac{c}{r} = 1 + e\cos\theta$ rolls on a straight line; the path of the focus is given by the equation

$$e^2 = 1 - \frac{2c}{y}\frac{dx}{ds} + \frac{c^2}{y^2};$$

and the path of the centre by

$$y^4\left(\frac{ds}{dx}\right)^2 = (a^2 + b^2)\,y^2 - a^2b^2,$$

$2a$ and $2b$ being the axes.

15. The roulette, on a straight line, of the centre of a rectangular hyperbola is

$$\frac{dx}{dy} = \frac{y^2}{\sqrt{a^4 - y^4}}.$$

16. A cycloid rolls on a straight line; the locus of its vertex is given by the equations

$$x = 2a(\sin\phi - \phi\cos\phi),\quad y = 2a\phi\sin\phi,$$

the origin being the point of the line over which the vertex passes.

17. A curve rolls symmetrically on an equal curve, carrying an involute; the envelope of this involute is an involute of the fixed curve.

18. If a curve roll on a straight line, the curvature of a point roulette varies as $\frac{d}{dp}\left(\frac{p}{r}\right)$, p and r being referred to the point.

If the curve be

$$\frac{a}{r} = 1 + \sec\alpha\sin(\theta\sin\alpha),$$

the roulette is a circle.

19. The curve $P'Q$ rolls on the curve PQ, P' passing over P; the roulette of P' is, in the neighbourhood of P, a semi-cubical parabola, of which the parameter is

$$\frac{9\rho\rho'(\rho+\rho')}{2(\rho+2\rho')^2},$$

ρ and ρ' being the radii of curvature at the point of contact.

20. A catenary, $s = c \tan \phi$, rolls symmetrically on an equal catenary; the intrinsic equation to the envelope of its axis is

$$\frac{d\sigma}{d\psi} = c \log \tan \frac{\pi - \psi}{4} + \frac{c}{2} \tan \frac{\psi}{2} \sec \frac{\psi}{2}.$$

21. If an oval curve roll on a straight line, prove that the area traced out by any point O in the curve will exceed the area of the curve by $\frac{1}{2}\int_0^{2\pi} r^2 d\phi$, where r is the distance from O of any point P of the curve, and ϕ the angle which the tangent at P makes with some fixed line in the curve: apply this to find the area of a cycloid.

22. If an oval curve A roll upon an equal and similar curve B, so that the point of contact is a centre of similitude for each, then the whole area traced out by any point O when A has made a complete revolution, is twice the area which would have been traced out if the curve A had rolled upon a straight line.

23. Test the formula of Art. (40) by applying it to an ellipse, measuring r from the focus.

24. A parabola rolls on a straight line from one end of the latus rectum to the other; the length of the arc enveloped by the axis is

$$2a \log (2\epsilon).$$

25. A parabola rolls symmetrically on an equal parabola, from one end of the latus rectum ($4a$) to the other; the length of arc enveloped by the axis is

$$2a \log (4\epsilon).$$

26. A diameter of a circle rolls on a curve; the envelope of the carried circle consists of two involutes of the curve.

27. A circle, radius b, rolls on a fixed circle of radius a; the area between the fixed circle, and the envelope of a diameter for a half roll from one end of the diameter to the other is equal to

$$\frac{\pi b^2}{4a}(3a+b).$$

28. The lemniscate $r^2 = a^2 \cos 2\theta$ rolls on a straight line; the tangential polar equation of the roulette produced by its pole is

$$\frac{dp}{d\phi} + p \tan \phi = a \tan \phi \sqrt{\sin \phi};$$

and the intrinsic equation to the envelope of its axis is

$$\frac{ds}{d\phi} = \frac{a}{6\sqrt{\cos \frac{2\phi}{3}}}\left(5 \sin \phi - 3 \sin \frac{\phi}{3}\right).$$

29. A circle rolls on a fixed circle; the envelope of any carried straight line is an involute of an epicycloid.

30. A catenary rolls on a straight line; the envelope of any carried straight line is an involute of a parabola.

31. An ellipse rolls, symmetrically, on an equal ellipse; prove that the whole length of the arc enveloped by its axis is

$$2a\left(1 + \frac{1-e^2}{e} \log \frac{1+e}{1-e}\right).$$

32. A curve, carrying a point, rolls on a straight line, and then, symmetrically, on an equal curve; prove that after rolling over the same arc in each case, the radii of curvature of the roulettes, and the distance of the point from the point of contact, are in Harmonic Progression.

33. In the same case, if a straight line be carried, the radii of curvature of the roulettes, and the distance of the line from the point of contact, are in Arithmetic Progression.

34. If a given arc of a curve roll, first externally, and then internally, over the same arc of a fixed curve, the sum or difference of the arcs of the roulettes of the same point is independent of the nature of the fixed curve; the sum, when the radius of curvature of the fixed curve, at each point of contact, is greater than that of the moving curve, and the difference, when the reverse is the case.

The same independence also exists for the sum, or difference, of the areas swept over by the straight line joining the carried point with the point of contact.

35. A parabola, latus rectum $4a$, rolls on a circle of radius b, the rolling commencing at the vertex. Prove that when the parabola rolls to the end of the latus rectum the corresponding arc of the roulette enveloped by the axis is

$$a\,(1 + \log 2) + \frac{4a^2}{3b}\,(2\sqrt{2} - 1).$$

36. An ellipse rolls on a straight line; the length of the envelope of its axis between two consecutive cusps is

$$2a\left(1 + \frac{1-e^2}{2e}\log\frac{1+e}{1-e}\right).$$

37. Find the envelope roulette of the directrix of an ellipse which rolls on a straight line; and prove that it has two cusps if the eccentricity is greater than $\frac{1}{2}(\sqrt{5} - 1)$, and that, if $e < \frac{1}{2}(\sqrt{5} - 1)$, the length of the arc of the roulette, corresponding to a complete roll of the ellipse, is equal to the perimeter of a circle, the radius of which is equal to the distance between the directrices of the ellipse.

38. The envelope roulette, on a straight line, of the axis of a rectangular hyperbola, is given by the intrinsic equation

$$s + a = \frac{a}{\sqrt{2}}\log\frac{\sqrt{2}+1}{\sqrt{2}\cos\phi + \sqrt{\cos 2\phi}} + \frac{a\cos\phi}{\sqrt{\cos 2\phi}}.$$

GLISSETTES.

57. GLISSETTES are the curves traced out by points, or enveloped by curves, carried by a curve, which is made to slide between given points or given curves.

Thus if an ellipse slide between two fixed straight lines at right angles to each other, the glissette traced by its centre is the arc of a circle.

Again, if a straight line, of given length, slide between two fixed straight lines at right angles to each other, the glissette of any point in the line is an ellipse.

In this case, if p be the perpendicular from the intersection of the fixed lines on the sliding line (length $2a$), and ϕ its inclination to one of the fixed lines,

$$p = a \sin 2\phi;$$

the envelope-glissette is therefore a four-cusped hypocycloid.

58. *A curve slides between two straight lines at right angles; to find the glissette of any carried point.*

Let the tangential polar equation of the curve, referred to the carried point, be

$$p = f(\phi);$$

then, if x, y be the perpendiculars from the carried point on the two fixed tangents,

$$x = f(\phi), \text{ and } y = f\left(\phi + \frac{\pi}{2}\right),$$

and the elimination of ϕ will give the rectangular equation of the glissette.

4—2

If the two fixed lines be not at right angles, but inclined to each other at an angle $\pi - \alpha$, and if x, y be the oblique co-ordinates of the carried point, we shall have to eliminate ϕ between the equations

$$x \sin \alpha = f(\phi), \quad y \sin \alpha = f(\phi + \alpha).$$

Ex. *An ellipse slides between two straight lines inclined to each other at an angle $\pi - \alpha$; to find the path of the centre.*

In this case, we can write the equations in the form

$$x^2 \sin^2 \alpha = a^2 \cos^2 \left(\phi - \frac{\alpha}{2}\right) + b^2 \sin^2 \left(\phi - \frac{\alpha}{2}\right),$$

$$y^2 \sin^2 \alpha = a^2 \cos^2 \left(\phi + \frac{\alpha}{2}\right) + b^2 \sin^2 \left(\phi + \frac{\alpha}{2}\right),$$

and the result of the elimination is

$$\{(x^2+y^2) \sin^2 \alpha - a^2 - b^2\}^2 + (x^2 - y^2)^2 \sin^2 \alpha \cos^2 \alpha = (a^2 - b^2)^2 \cos^2 \alpha.$$

59. *The following theorem is of great importance.*

Any state of motion of a plane area in its own plane can be represented by a state of rotation about a point.

Any plane area is fixed if two of its points are fixed; and, if the motions of two of its points are given, the motion of the area is given.

We must observe, however, that we cannot assign an arbitrary motion to two points; the restriction existing that the velocities of the two points in the direction of the line joining them must be the same.

Suppose that two points P and Q are in motion in the directions PT and QV, and that PE, QE are drawn perpendicular to those lines respectively, and meeting in E.

It is clear that the motion of P may be represented by a state of rotation about any point in the line PE, and that of Q by a rotation about any point in QE.

Hence, both motions are represented by a state of rotation about E.

This point E is called the *instantaneous centre of rotation*, and the motion of any point R in the area is, at the instant considered, at right angles to RE.

Thus, if a curve slide between two given lines, the intersection of the normals at the points of contact is the instantaneous centre.

60. *Any motion of an area in its own plane can be represented by the rolling of a certain determinate curve on another determinate curve.*

Let $PQRS$ (fig. 37) be the curve traced out in space by the successive positions of the instantaneous centre, and let the angular velocities of the moving area corresponding to each position of the instantaneous centre be known.

Then, if P, Q, R, ... be successive positions of the centre at given infinitesimal intervals of time, the lines in the moving area QP, RQP', $SRq'p'$, ... will turn successively into the positions QP', $Rq'p'$, $Srqp$, &c.

Hence the motion can be represented by rolling the curve $pqrs$ ... on the curve $PQRS$ ....

But the curve $pqrs$ ... is the locus on the moving area of the instantaneous centre, and the two curves are therefore determinate.

These curves are sometimes called the *fixed and moving centrodes.*

61. Ex. 1. *A straight line* AB, *of given length, slides between two fixed straight lines at right angles to each other.*

In this case the locus of the instantaneous centre, E, with regard to the fixed lines OX, OY, is the circle, centre O and radius OE, which is equal to AB, fig. 38, while the locus of E with regard to AB is the circle on AB as diameter.

The motion is therefore the same as that produced by the rolling of a circle, internally, on a fixed circle of double its radius.

The effect of this motion is discussed in Art. (7), and therefore it follows that the path of any point Q, connected with AB, is an ellipse of which O is the centre.

It has been shewn in Art. (57), *that the envelope of the sliding line is a four-cusped hypocycloid.*

We can prove this result also by direct geometry.

For if QCQ', fig. 39, is the sliding line, and if EP is the perpendicular from E upon QCQ', it follows that P is the point of contact of the envelope of QQ', and therefore that the locus of P is the envelope.

It is clear that P is a point on the circle whose diameter is CE, and CE is half of OE, and is therefore one-fourth of the diameter of the fixed centrode EA.

Further the arc EP of the small circle is equal to the arc EQ of the rolling centrode, and therefore equal to the arc EA of the fixed centrode.

Hence it follows that the path of P is the path of a point in the perimeter of a circle rolling inside a circle of four times its radius, and is therefore a four-cusped hypocycloid.

Further we can see that the envelope of any straight line through C, making a given angle with QQ' and carried with it, will be a four-cusped hypocycloid.

For the ends of this new diameter of the rolling centrode will move along fixed radii of the fixed centrode, and the new diameter will be under the same conditions as QQ'.

If any other straight line, not passing through C *be carried by* QQ′ *the envelope-glissette will be an involute of a four-cusped hypocycloid.*

Let the line be at the distance c from C, and inclined at the angle α to QQ', and let p be the perpendicular from O upon the line and ϕ the inclination of the perpendicular to OA.

Then $2a$ being the length of the sliding line

$$p = c + a \sin(2\phi + \alpha),$$

$\therefore$ the radius of curvature of the envelope

$$= c - 3a \sin(2\phi + \alpha),$$

so that the envelope is an involute of a four-cusped hypocycloid.

62. *The same results are true whatever be the angle between the two fixed lines.*

For if this angle is α, OE being the diameter of the circumscribing circle of QEQ' is equal to $2a \operatorname{cosec} \alpha$, and is therefore constant.

The fixed centrode is therefore the circle, centre O and radius $2a \operatorname{cosec} \alpha$, also since QEQ' is constant, the rolling centrode is the circle $QOQ'E$, the radius of which is $a \operatorname{cosec} \alpha$.

63. Ex. 2. *Two straight lines,* AB, AC, *containing a given angle, move so that* AB, AC *pass respectively through fixed points* P *and* Q.

The point E being the instantaneous centre, AE is the diameter of the circle about PAQ, and is constant. Fig. 40.

Hence, the moving centrode is the circle, centre A and radius AE, while the fixed centrode is the circle PAQ.

The sliding motion is therefore equivalent to the motion produced by rolling a circle, with internal contact, on a circle of half its radius.

The point-glissettes are therefore hypotrochoids, and the line-glissettes are circles.

This last can be shewn by the formula of Art. (14), for if EN be the carried line, at the distance CN from C, $OPQ = \alpha$, $\rho' = a$, and $\rho = -2a$, fig. 41;

$$\therefore \frac{d\sigma}{d\phi} = PQ + 2a \cos OCA' = CN,$$

i.e. the curvature is constant.

Or, by direct geometry we can see that if A roll from A', and G be the other end of the diameter $A'OG$,

$$GQ = CN,$$

and therefore the envelope of EN is a circle, centre G and radius CN.

If the straight lines AB, AC slide on fixed circles, the results are exactly the same.

For the straight lines drawn through the centres of the circles parallel to AB and AC meet at the same angle, and the sliding of these lines through the centres carries with it the motion of BAC.

64. Ex. 3. *An involute of a circle slides between two straight lines at right angles to each other.*

Let Ox, Oy be the fixed lines, fig. 42, and let x and y be co-ordinates of E, the instantaneous centre.

Then $$x=\left(\frac{\pi}{2}+\phi\right)a-a, \text{ and } y=a\phi+a;$$

so that $$y-x=\left(2-\frac{\pi}{2}\right)a.$$

In this case the fixed centrode is a straight line, and the moving centrode is a circle, centre C.

The point-glissettes are therefore cycloids and trochoids, and the line-glissettes are evolutes of a cycloid.

65. *A curve, carrying a point* Q, *slides on a fixed curve, the same point* P *of the moving curve being always in contact with the fixed curve.*

In this case the instantaneous centre is the centre of curvature E at P, and therefore QE is the normal to the Q-glissette.

If the curve slides over the arc PP' of the fixed curve, and Q moves over the elemental arc QQ', and if E' is the centre of curvature at P', $Q'E'$ is the consecutive normal, and F is the centre of curvature at Q. Fig. 43.

If $PEP'=\delta\phi$, $PE=\rho$, $PQ=c$, and if α is the inclination of PQ to the normal at P,

$$QFQ'=\delta\phi-\frac{EE'\sin PEQ}{QE}=\delta\phi-\delta\rho\,\frac{c\sin\alpha}{QE^2}.$$

Let $\rho' =$ radius of curvature at Q,

and $\rho'' =$ radius of curvature of the evolute at E; then

$$\frac{1}{\rho'} = \frac{QFQ'}{QQ'} = \frac{QFQ'}{QE\delta\phi}$$

$$= \frac{1}{QE} - \frac{d\rho}{d\phi}\frac{c\sin\alpha}{QE^3};$$

i.e. $$\frac{1}{\rho'} = \frac{1}{(\rho^2 + c^2 + 2c\rho\cos\alpha)^{\frac{1}{2}}} - \frac{\rho'' c\sin\alpha}{(\rho^2 + c^2 + 2c\rho\cos\alpha)^{\frac{3}{2}}}.$$

In this case the fixed centrode is the evolute of the fixed curve and the moving centrode is the normal at P, so that the motion is equivalent to the rolling of a straight line on the evolute.

If the sliding curve be a straight line, and the point Q a point in the sliding line, the angle α is a right angle, and the curvature of Q is given by the equation

$$\frac{1}{\rho'} = \frac{1}{(\rho^2 + c^2)^{\frac{1}{2}}} - \frac{\rho'' c}{(\rho^2 + c^2)^{\frac{3}{2}}}.$$

66. *To find the point-glissettes and the line-glissettes, when a curve slides so as to touch a fixed straight line at a fixed point.*

P being the carried point, the relation $OP = f(PY)$ is known, and therefore if $OY = x$ and $PY = y$, fig. 44,

$$\sqrt{x^2 + y^2} = f(y),$$

is the point-glissette.

Again, if AP is the carried line, and OZ the perpendicular upon it from O, let

$$OZ = p \text{ and } ZOY = \psi.$$

Then, if $TAP = \alpha$, and $s = f(\phi)$ is the intrinsic equation of the curve referred to A and the tangent at A

$$\alpha = \phi + \frac{\pi}{2} - \psi$$

and $$p = \int ds \sin\phi = \int f'(\phi)\sin\phi d\phi.$$

From these equations we can obtain p in terms of ψ, *i.e.* the tangential polar equation of the line-glissette.

In this case the instantaneous centre E is the centre of curvature at O, so that the fixed centrode is the straight line OE, and the moving centrode is the evolute of the curve.

67. *A given point* A *of a straight line moves along a fixed line* Ox, *and the straight line passes through a fixed point* C, *at the distance* CO, (c), *from* Ox.

The instantaneous centre, E, is the point of intersection of the straight line through C, perpendicular to AC, and of the straight line through A perpendicular to Ox.

Taking x, y as the co-ordinates of E referred to Ox and OC, we observe that if θ is the inclination of CA to OA,

$$x = c \cot \theta, \text{ and } y = c + x \cot \theta,$$

so that the equation of the fixed centrode is

$$x^2 = c(c - y).$$

Again if x', y' are the co-ordinates of E referred to AC and the perpendicular through A to AC,

$$c = x' \sin \theta, \text{ and } x' = y' \tan \theta,$$

so that the moving centrode is the curve

$$c^2y^2 = x^2(x^2 - c^2),$$

and the motion is equivalent to the rolling of this curve on the parabola, $x^2 = c^2 - cy$.

68. *A straight line passes through a fixed point* O, *and a given point* A *of the line moves along a given fixed curve.*

Taking O for the origin, let $r = f(\theta)$ (fig. 45), be the equation of the given curve.

Then, E being the instantaneous centre if $OE = r'$, and $EOy = \theta'$, r', θ' are co-ordinates of a point of the fixed centrode.

Now $\qquad \theta' = \theta, \text{ and } r' = r \tan \phi = \dfrac{dr}{d\theta} = f'(\theta),$

$\therefore$ the fixed centrode is $r' = f'(\theta')$.

Again, if $AE = \rho$,

$$\rho = r \sec \phi = \sec \phi f(\theta),$$

$$\tan \phi = \frac{1}{r}\frac{dr}{d\theta} = \frac{f'(\theta)}{f(\theta)},$$

and the moving centrode will be obtained by the elimination of θ.

69. *A given point* A *of a straight line moves along a fixed line* Ox, *and the moving straight line always touches a given fixed curve.*

The point E being the instantaneous centre, let $AE = \rho$ and $EAP = \phi$, fig. 46, then x, y being the co-ordinates of P, the point of contact,

$$\rho = AP \sec \phi = y \sec^2 \phi = y \left\{1 + \left(\frac{dx}{dy}\right)^2\right\},$$

and $$\cot \phi = -\frac{dy}{dx}.$$

Hence eliminating x and y, we obtain the polar equation of the moving centrode.

For the fixed centrode,

$$\xi = OA = x + y \tan \phi = x - y\frac{dx}{dy},$$

$$\eta = AE = y \left\{1 + \left(\frac{dx}{dy}\right)^2\right\},$$

and the elimination of x and y will give the Cartesian Equation of the fixed centrode.

70. *Holditch's Theorem.*

If a straight line APB, *of given length* a + b, *moves with its ends* A *and* B *on the arc of a closed oval curve, and if* AP = a, *and* BP = b, *the difference between the area of the oval, and the area of the locus of the point* P, *is equal to* πab.

Let Q be the point of intersection of AB with a consecutive position inclined to AB at the angle $\delta\theta$.

Then if U be the area of the oval, V the area of the locus of P, and W the area of the locus of Q, that is, of the envelope of AB, the difference $U - W$ is equal to $\frac{1}{2}\int_0^{2\pi} AQ^2 d\theta$, and is also equal to $\frac{1}{2}\int_0^{2\pi} (a+b-AQ)^2\, d\theta$.

Hence, if $AQ = r$,

$$\int_0^{2\pi} r d\theta = \frac{1}{2}\int_0^{2\pi} (a+b)\, d\theta = \pi\,(a+b),$$

also
$$V - W = \frac{1}{2}\int PQ^2 d\theta = \frac{1}{2}\int (a-r)^2\, d\theta,$$

$$\therefore\ U - V = \frac{1}{2}\int \{r^2 - (a-r)^2\}\, d\theta = \frac{1}{2}\int (2ar - a^2)\, d\theta$$
$$= a\pi\,(a+b) - \pi a^2 = \pi ab.$$

This proof was given in the *Quarterly Journal of Mathematics*, vol. 2, 1858.

71. *Amsler's Planimeter.*

Amsler's Planimeter is an instrument employed to determine practically the area within any closed curve on a plane surface, as, for instance, the area of an estate marked out on an ordnance map.

It consists of two straight rods, AB, BC, jointed at B, and capable of free motion about the end A, which is fixed, with a small wheel, the plane of which is perpendicular to BC, capable of turning round BC as an axis.

The centre of the wheel is sometimes at the end B of BC, and sometimes at some point P between B and C, but it might be placed at any point of BC produced either way.

The important point to be observed is that the plane of the wheel is perpendicular to BC, so that the revolution of the wheel determines the motion of the point B in the direction perpendicular to BC.

We will first take the case in which the fixed point A is outside of the contour line enclosing the area to be measured.

Let $AB = a$, $BC = b$, and let ϕ and ψ be the inclinations of AB and BC to some fixed direction in the plane of motion. Fig. (47).

When the wheel, radius c, is at B, let the end C pass over an elemental arc CC' of the contour line, so that ABC takes up the position $AB'C'$.

If $d\theta$ is the angle turned through by the wheel, the area

$$ABCC'B'A = \frac{1}{2}a^2 d\phi + \frac{1}{2}b^2 d\psi + b\,.\,cd\theta;$$

for, neglecting infinitesimals of the second order, it is made up of the triangle ABB', the parallelogram $BCDB'$, $B'D$ being parallel and equal to BC, and the triangle $B'DC'$. Fig. (48).

When C has moved completely round the contour line, and has returned to its initial position, the complete area swept over by ABC will be the area enclosed by the contour line, and this

$$= \int\left(\frac{1}{2}a^2 d\phi + \frac{1}{2}b^2 d\psi + bcd\theta\right)$$

$$= bc\theta = bs,$$

where s is the total length passed over by the point B in the direction always perpendicular to BC.

When the centre of the wheel is at P, let $d\theta_1$ be the angle turned through by the wheel while AB turns through $d\phi$, BC moving parallel to itself into the position $B'D$, and $d\theta_2$ the angle turned through while $B'D$ turns through $d\psi$ into the position $B'C'$.

Then $cd\theta_1$ is the distance between the parallel lines BC, $B'D$, and $cd\theta_2$ is the distance QP' perpendicular to $B'Q$.

Therefore the area $ABCC'B'A$

$$= \frac{1}{2}a^2 d\phi + \frac{1}{2}b^2 d\psi + bcd\theta_1$$

$$= \frac{1}{2}a^2 d\phi + \left(\frac{1}{2}b^2 d\psi - bcd\theta_2\right) + bc\,(d\theta_1 + d\theta_2).$$

But $$cd\theta_2 = QP' = ld\psi, \text{ if } BP = l,$$

$\therefore$ the area $ABCC'B'A$

$$= \frac{1}{2}a^2d\phi + \left(\frac{1}{2}b^2 - bl\right)d\psi + bds,$$

if ds is the element of the distance passed over by P in the direction perpendicular to BC.

Hence, as before, the area enclosed by the contour line

$$= bs,$$

where s is the total length traversed by P in the direction always perpendicular to BC.

We will now consider the case in which the point A is inside the contour line.

Adopting the same notation as in the previous case,

The area enclosed by the contour line, fig. (49),

$$= \int\left\{\frac{1}{2}a^2d\phi + bc\,(d\theta_1 + d\theta_2) + \left(\frac{1}{2}b^2 - bl\right)d\psi\right\}$$

$$= \pi\,(a^2 + b^2 - 2bl) + bs.$$

If the point B moves outside the closed curve, as in fig. (50), the area enclosed is the difference between the areas swept over by AB and BC; i.e. the area

$$= \int\left\{\frac{1}{2}a^2d\phi - \frac{1}{2}b^2(-d\psi) - b\,(-cd\theta_1)\right\}.$$

Observing that θ_1 and θ_2 are now registered negative, and that BC is also turning in the negative direction, we have the relation,

$$c\,(-d\theta_2) = l\,(-d\psi),$$

$$\therefore \text{ the area } = \int\left(\frac{1}{2}a^2d\phi + \frac{1}{2}b^2d\psi + bcd\theta_1\right)$$

$$= \int\left\{\frac{1}{2}a^2d\phi + \frac{1}{2}b^2d\psi - bcd\theta_2 + bc\,(d\theta_1 + d\theta_2)\right\}$$

$$= \int\left\{\frac{1}{2}a^2d\phi + \left(\frac{1}{2}b^2 - bl\right)d\psi + bds\right\},$$

ds being the element of the distance traversed by P in the direction perpendicular to BC.

Hence as before the expression for the area is

$$\pi (a^2 + b^2 - 2bl) + bs.$$

72. *A given curve slides between two straight lines at right angles to each other; to find the locus, with regard to the lines, of the instantaneous centre.*

Let x, y be co-ordinates of E, C the centre of curvature at P, and CK an element of the evolute at P (fig. 51).

Then, by turning the curve round E through a small angle $\delta\phi$, and taking K such that the angle of deflection of the arc CK is $\delta\phi$, the tangent at K will glide into the position $K'P'$, and we have

$$\delta x = -PP' = -(\rho - y)\,\delta\phi;$$

$$\therefore\ \frac{dx}{d\phi} - y = -\rho.$$

Similarly, we shall obtain

$$\frac{dy}{d\phi} + x = \rho',$$

if ρ' be the radius of curvature at Q.

Hence, if the intrinsic equation be known,

$$\rho = f(\phi) \text{ and } \rho' = f\left(\phi + \frac{\pi}{2}\right),$$

and the two equations we have obtained will give x and y in terms of ϕ, and reduce the solution of the problem to the elimination of ϕ.

Take, for instance the case of Art. (64), in which

$$\rho = a\phi,$$

and therefore
$$\frac{dx}{d\phi} - y = -a\phi,$$
$$\frac{dy}{d\phi} + x = a\left(\frac{\pi}{2} + \phi\right).$$

Integrating these equations, and remembering that when

$$\phi = 0,\ x = \frac{\pi a}{2} - a, \text{ and } y = a,$$

we shall obtain the equations of Art. (64).

73. The preceding equations determine the tangent and normal at E, and, by differentiation, the curvature of the locus of E.

A geometrical construction may however be given, for since $EN = PP' = EC\delta\phi$, and $E'N = EC'\delta\phi$ (fig. 52),

$$\therefore EN : E'N :: EC : EC',$$

and the angle
$$NEE' = ECC'.$$

Hence, if EF be perpendicular to EE',

the angle
$$FEC' = \frac{\pi}{2} - E'EN = EC'F,$$

and F is the middle point of CC'.

If then D, D' be the centres of curvature consecutive to C, C' in their new positions, and F' the middle point of DD', the lines EF, $E'F'$ intersect in the centre of curvature at E.

For the carried locus, i.e. the locus of E with regard to the sliding curve, we observe that the tangent and normal are the same as for the locus above considered.

For the curvature, however, let CK, $C'K'$, fig. 53, be elements of the evolute having the same deflection $\delta\phi$, E' the intersection of the tangents at K and K', and F' the middle point of KK'; then G, the intersection of EF and $E'F'$, is the centre of curvature.

In either case, CK being an infinitesimal of the first order,

$$\text{arc } EE' = EF \,.\, 2\delta\phi = CC'\delta\phi,$$

or ultimately
$$\frac{d\sigma}{d\phi} = CC'.$$

Also, if the FEC'' in (figure 52) $= \psi$,

$$\tan\psi = \frac{EC}{EC''},$$

and, if EC and EC'', i.e. $\rho - y$ and $\rho' - x$, can be found in terms of ϕ, we shall have sufficient data for the determination of $\frac{d\sigma}{d\phi}$, the radius of curvature.

For the carried locus (fig. 53), we must add to ψ the inclination of EC' to some fixed line in the moving area.

74. *If a right angle slide on the arc of a curve,* $p = f(\phi)$, *the motion is equivalent to that of rolling the curve*

$$x = f\left(\phi + \frac{\pi}{2}\right) - f'(\phi),\ y = f(\phi) + f'\left(\phi + \frac{\pi}{2}\right),$$

upon the curve

$$x' = \cos\phi f'\left(\phi + \frac{\pi}{2}\right) + \sin\phi f'(\phi),$$

$$y' = \cos\phi f'(\phi) - \sin\phi f'\left(\phi + \frac{\pi}{2}\right).$$

For (fig. 54) if P and Q be the points of contact, the intersection E, of the normal is the instantaneous centre, and if $TP = x$, $TQ = y$, x and y are the coordinates of the carried locus of E, and since

$$OY = p = f(\phi),$$

and
$$PY = \frac{dp}{d\phi} = f'(\phi),$$

$$TY = f\left(\phi + \frac{\pi}{2}\right) \text{ and } QY' = f'\left(\phi + \frac{\pi}{2}\right).$$

Hence the first two equations follow at once.

For the fixed locus of E let fall the perpendicular EN on the initial line from which ϕ is measured; then if $ON = x'$ and $EN = y'$, the second pair of equations is obtained.

As an example, let the given curve be a three-cusped hypocycloid, so that

$$p = a \cos 3\phi.$$

Then it will be found that

$$x^2 + y^2 = 16a^2 \text{ and } x'^2 + y'^2 = 9a^2,$$

so that the motion is produced by rolling a circle of radius $4a$, with internal contact, on a circle of radius $3a$.

This result can be obtained from direct Geometry.

For (fig. 55) if F, G, be two positions of the centre of the rolling circle on a diameter QOQ', the tangents TP, TP' at the points P and P' of the curve will be at right angles; and the normals QP, $Q'P'$ will meet at a point E on the circumference of the fixed circle. It will be easily seen that the line TE passes through O, and is four times the radius (c) of the rolling circle, so that while the fixed locus of E is the circle of radius $3c$, the carried locus is a circle of radius $4c$, and having T for its centre.

It will be seen that the equations of Art. (72), i.e.

$$\frac{dx}{d\phi} - y = -\rho, \quad \frac{dy}{d\phi} + x = \rho';$$

follow at once from the preceding equations.

75. Taking the general case of any motion of a plane area (which however is reducible to a case of roulettes), and supposing that the centrodes can be found, or, which is the same thing, the fixed centrode, and the rates of rotation as compared with the increase of arc along the centrode, we can find simple expressions for the curvatures of the roulettes.

To find the curvature of the roulette of a point Q.

Let EE' ($\delta\sigma$) be an element of the fixed centrode fig. 56, and α the inclination of EQ (r) to the normal at E.

Then $$VQ : EQ :: QQ' : QQ' - E'N,$$
if $E'N$ is perpendicular to EQ;
or radius of curvature $= VQ$

$$= \frac{r^2 d\phi}{r d\phi - d\sigma \,.\, \cos\alpha}.$$

If we take the case of Art. (34),

$$d\phi = d\sigma\left(\frac{1}{\rho} + \frac{1}{\rho'}\right);$$

and we obtain the formula of that article.

If we take the case of Art. (73),

$$d\sigma = CC'\delta\phi,$$

and $\therefore$ radius of curvature $= \dfrac{r^2}{r - \cos\alpha \,.\, CC'}$.

76. *To find the curvature of an envelope-roulette.*

Let $r = EQ$ be the perpendicular from E on the line fig. 57.

Then $$\delta s = Qq + qQ' = \delta\sigma\cos\alpha + EQ\delta\phi,$$

and $$\frac{ds}{d\phi} = r + \cos\alpha\,\frac{d\sigma}{d\phi}.$$

For instance in the case of Art. 73, the radius of curvature

$$= r + \cos\alpha \,.\, CC'.$$

77. *To find the curvature of the envelope of any carried curve.*

Let EQ, $E'q$ be normals to the carried curve, and ρ'' the radius of curvature at Q, fig. 58.

Then $$\delta s = Qq + Q'q$$

$$= \delta\sigma\cos\alpha\,\frac{\rho''}{\rho'' + EQ} + EQ\delta\phi,$$

and if $$QVQ' = \delta\psi,$$

$$\delta\phi - \delta\psi = E'OE = \frac{\delta\sigma\cos\alpha}{\rho'' + r};$$

therefore, radius of curvature of envelope

$$= \frac{ds}{d\psi}$$

$$= \frac{\dfrac{\rho'' \cos\alpha}{\rho'' + r} d\sigma + r d\phi}{d\phi - \dfrac{\cos\alpha\, d\sigma}{\rho'' + r}}.$$

For instance, taking the case of Art. (50), we have

$$d\phi = d\sigma\left(\frac{1}{\rho} + \frac{1}{\rho'}\right),$$

and we fall upon the formula of that article.

78. *A triangle moves in a plane so that two of its sides slide on fixed curves; it is required to find the envelope of the third side.**

Let α, β, γ be the perpendiculars from any fixed point on the sides a, b, c; then

$$a\alpha + b\beta + c\gamma = 2 \text{ (area of triangle)} = 2\Delta.$$

Hence, if ϕ be the angle which one of the perpendiculars makes with any fixed line, and remembering that

$$\alpha + \frac{d^2\alpha}{d\phi^2}$$

is an expression for the radius of curvature of the envelope of the side BC, we obtain

$$a\rho_1 + b\rho_2 + c\rho_3 = 2\Delta,$$

where ρ_1, ρ_2, ρ_3 are the radii of curvature of the envelopes of the several sides.

If then two of these be given, the third is determined.

79. Ex. 1. *A given triangle moves so that two of its sides touch fixed circles.*

In this case ρ_1 and ρ_2 are constant; ρ_3 is therefore constant, and the third side always touches a fixed circle.

* The method of this article is due to Dr Ferrers.

This includes the example of Art. (63) as a particular case.

A direct geometrical proof may be also given*.

Firstly, let the sides AB, AC pass through fixed points P, Q.

Through A draw AF parallel to BC, and meeting in F the fixed circle which is the locus of A, fig. 59.

The angle $FAP = ABC$, and therefore the arc FP is constant, and F is a fixed point.

Also, the perpendicular FG from F on BC is equal to the altitude of the triangle ABC; therefore BC always touches a circle, the centre of which is at F.

Secondly, let AB, AC touch fixed circles having their centres at P' and Q'.

Through P' and Q' draw lines $P'A'$, $Q'A'$ parallel to AB, AC, and meeting BC in B' and C': then, as before, $B'C'$ touches a fixed circle.

80. Ex. 2. *A triangle moves so that two of its sides slide on the arc of a fixed cycloid.*

In this case

$$\rho_1 = d \sin \phi, \text{ and } \rho_2 = d \sin (\phi + \alpha);$$

$$\therefore c\rho_3 = 2\Delta - ad \sin \phi - bd \sin (\phi + \alpha),$$

which can be written in the form

$$\rho_3 = e + f \sin (\phi + \beta).$$

Hence it appears that the envelope of the third side is an involute of a cycloid.

81. *Two straight lines* AB, AC, *inclined to each other at a given angle* λ, *and carrying a line* AD, *slide on fixed curves; it is required to find the envelope of* AD.

This is a particular case of Art. (78), and if μ, ν be the

* Several proofs were given, by myself and others, in the *Educational Times* for 1864.

inclinations of AD to AB and AC, the first equation of that article becomes

$$\alpha \sin \mu + \beta \sin \nu + \gamma \sin \lambda = 0.$$

Hence $$\rho_1 \sin \mu + \rho_2 \sin \nu + \rho_3 \sin \lambda = 0,$$

and if ρ_1 and ρ_2 are given, ρ_3 is determined.

82. Ex. 1. Thus, from Ex. (2), Art. (79), we obtain the following result.

If two straight lines at right angles to each other slide on the arc of a cycloid, the straight line bisecting the angle between them always touches a cycloid.

Ex. 2. AB, AC *slide completely round an oval curve.*

In this case, if

$$-\rho_3 = \frac{d\sigma}{d\phi},$$

$$\sin \lambda \frac{d\sigma}{d\phi} = \sin \mu \frac{ds}{d\phi} + \sin \nu \frac{ds'}{d\phi},$$

ds, ds' being elements at the points of contact of AB, AC.

Hence, if l be the perimeter of the oval, and σ the whole arc enveloped by AD,

$$\sin \lambda \,.\, \sigma = l (\sin \mu + \sin \nu),$$

or $$\sigma = l \frac{\sin \frac{\mu + \nu}{2}}{\sin \frac{\mu - \nu}{2}}.$$

We have put

$$\frac{d\sigma}{d\phi} = -\rho_3,$$

because if α and β are positive, γ is necessarily negative.

EXAMPLES.

1. P is a fixed point on the circumferencc of a given circle, and PQ any chord drawn through P is produced to R, so that QR is of constant length. If PE be drawn perpendicular to QP to cut the circle again in E, and RE be joined, shew that RE is normal to the locus of R.

2. Prove that a limaçon, $r = a + b\cos\theta$, is a point roulette produced by the rolling of a circle, with internal contact, on a fixed circle of half its radius.

3. The angle BAC slides over two fixed circles; prove that the point-glissettes are limaçons.

4. A parabola slides on two straight lines at right angles to each other; prove that its vertex and focus respectively describe the curves,

$$x^2y^2(x^2+y^2+3a^2)=a^6, \quad \text{and} \quad x^2y^2=a^2(x^2+y^2).$$

5. C is a fixed point external to a given circle whose centre is O. TEB is a tangent of given length, touching the circle in E; its extremity T being in CO produced. If CB and OE when produced intersect in M, shew that the length of BC is least when TM is at right angles to TC.

6. A plane moves in any (given) manner on a fixed plane: O is a fixed point on the fixed plane, P a fixed point on the moving plane; if the area described by P about O is given, shew that the locus of all points (P) in the moving plane for which this area is the same is a circle, and that for different values of the area the corresponding circles are concentric.

7. A given triangle moves in a plane so that one of its sides touches a fixed circle, and another a fixed cycloid; prove that the third side touches an involute of a cycloid.

8. Two tangents OP, OQ inclined at a constant angle α are drawn to a closed curve, without points of inflexion, and OX is the external bisector of the angle POQ. Shew that the perimeter of the closed curve enveloped by OX : the perimeter of the original curve :: 1 : $\sin \frac{1}{2}\alpha$.

9. One end A of a straight line slides along a fixed straight line OA, and the straight line always passes through a fixed point C at the distance CO, (c), from the line OA; prove that this motion is equivalent to the rolling of the curve, $c^2(x^2+y^2)=x^4$, upon the parabola, $cy=x^2+c^2$.

10. If a parabola, latus rectum $4a$, slide between two straight lines at right angles to one another, the glissettes produced are the same as the roulettes produced by a parabola latus rectum a, rolling on the curve

$$a^2(x^2+y^2)^3=x^4y^4.$$

11. A straight line slides on a curve having always the same point in contact; the motion is identical with that of rolling a perpendicular straight line on the evolute.

Hence shew that the envelope of the straight lines drawn through each point of an epicycloid at a constant angle to the tangent is also an epicycloid.

12. An ellipse slides on a straight line, always touching it at the same point; the path of its centre is the curve

$$x^2y^2=(a^2-y^2)(y^2-b^2).$$

13. A straight rod, of length $2a$, slides with its ends on a wire in the form of the cardioid, $r=a(1-\cos\theta)$; prove that the fixed and moving centrodes are circles.

14. A straight line ACB slides on a fixed curve, the middle point C being always in contact with the curve; if $1/\rho$ is the curvature at C, and $1/r$ and $1/s$ are the curvatures of the paths of A and B, and if $AB=2c$, prove that

$$1/r+1/s=2/\sqrt{\rho^2+c^2}.$$

15. A given point in a straight line moves along a given diameter, produced, of a given circle, and the straight line always touches the circle; prove that this motion is equivalent to the rolling of a parabola upon the curve,

$$a^2y^2 = (x+a)^2 (2ax + x^2),$$

the parabola starting with its vertex at the origin.

16. The end P of a finite line PQ travels on a closed curve which has no points of inflexion, and the line PQ is inclined at a constant angle to the normal at P. Find the area included between the locus of Q and the curve, and shew that if O be the centre of curvature at P the motion of Q is perpendicular to QO.

17. Two tangents, inclined to each other at a given angle, move round a closed curve without cusps, or points of inflexion, and the external angle is divided into two constant parts α and β; prove that the length of the envelope of the dividing line is to the length of the given curve, as

$$\cos\frac{\alpha-\beta}{2} \text{ is to } \cos\frac{\alpha+\beta}{2}.$$

18. One end of a straight rod moves round the circumference of a circle, and the rod always passes through a fixed point of the circumference; prove that this motion can be produced by rolling a circle with internal contact upon a circle of half its radius.

19. If a straight rod pass through the vertex of a parabola, and one end move along the arc of the curve, shew that this motion can be produced by rolling the curve $a^2y^4 = (x^2 + 4a^2)\,\{x^2 + 4a^2(x^2 + y^2)\}$ upon the curve $8ay^2 = x^2(x - 4a)$.

20. An involute of a circle slides on a straight line, always touching it at the same point; the glissettes of a point and a straight line are respectively a trochoid and an involute of a cycloid.

21. A parabola slides on a straight line touching it at a fixed point P.

If the normal at P meet the axis in G and GR be drawn parallel to SP and equal to one fourth of the latus rectum, the normal to the path of the focus is parallel to PR.

Shew also that the path of the focus is an hyperbola.

22. If at any point of a curve whose intrinsic equation is $s = f'(\phi)$ a straight line is drawn making a constant angle α with the tangent and of length $s \cos \alpha$, the intrinsic equation of the locus of its extremity will be

$$s = f'(\phi) \sin \alpha \pm f(\phi) \cos \alpha.$$

23. A fixed point of a straight line moves along the axis, produced, of a parabola, and the straight line always touches the parabola; prove that the motion is equivalent to the rolling of the curve,

$$4a^2y^4 = x^2(x^2 - 4a^2y^2),$$

upon the curve, $ay^2 = 4x(x+a)^2$.

24. A lamina moves in its own plane so that a point fixed in it lies on a straight line fixed in the plane, and that a straight line fixed in it always passes through a point fixed in the plane; the distances from each point to each line being equal. Prove that the fixed and moving centrodes are parabolas.

25. A given right-angled triangle PQR is made to slide round the outside of a fixed oval curve with the point P on the curve, the side PR touching it, and the side PQ normal to it. If s is the perimeter of the oval, prove that the length of the curve enveloped by QR is equal to

$$(s + 2\pi \,.\, PQ) \sin PQR.$$

GENERAL MOTION OF A RIGID BODY.

83. A rigid body is said to have a motion of *translation*, when all planes in the body move parallel to themselves; or, which comes to the same thing, when all points of the body pass over equal distances in the same direction.

A rigid body is said to have a motion of *rotation*, or to have *rotatory* motion, when some plane in the body changes its angle of inclination to some plane fixed in space.

In general, the motion of a rigid body can always be represented by a rotation combined with a translation, and the translation may be rectilinear or curvilinear.

84. The wheel of a carriage, for instance, has a rectilinear motion of translation, combined with a rotatory motion about its centre; but this, as we have seen, can be represented by a state of rotation about the point of contact of the wheel with the ground.

If a man looking straight at a particular wall of a room, walks round a table in the room, he has a motion of circular translation, but no motion of rotation; all the points of his body moving in equal circles with different centres.

As another illustration, the moon moves round the earth so as always to present very nearly the same face to the earth.

It follows therefore that, while the centre of the moon moves round the earth in its oval orbit, an ellipse of small eccentricity, the moon turns completely round an axis through its centre, that is to say, it has an angular velocity of four right angles per month*.

The motion of the moon is therefore represented by a motion of elliptic translation, combined with a motion of rotation.

Again, when a mass of liquid rotates uniformly, as if rigid, about a vertical axis, every molecule describes a circle, and has besides a rotatory motion, the free surface being a paraboloid; whereas, in Rankine's free circular vortex, every molecule describes a circle, but has no rotatory motion, and the free surface is convex, and has a horizontal asymptotic plane.

Motion of a rigid body about a fixed point.

85. If two straight lines OP, OQ, through the fixed point O, are fixed, it is clear that the body is incapable of motion, and the motion of the body is therefore completely determined by the motions of these two straight lines.

At any instant the point P is in motion in some definite direction, and the line OP has a motion in the plane containing this direction.

Drawing the plane POC through OP perpendicular to its plane of motion, the motion of OP can be represented by a state of rotation about any line through O in this perpendicular plane.

Similarly the motion of OQ can be represented by a state of rotation about an axis in a plane QOC, intersecting the other plane in the line OC. Both motions are represented by a single rotation about the line OC.

The motion of a rigid body about a fixed point is therefore, at any instant, a motion of rotation about some axis through the point.

* About thirty years ago a curious controversy took place in the columns of the 'Times', concerning the motion of the Moon. It was asserted that, because the Moon always presents the same face to the Earth, it has no rotatory motion, and there was a good deal of correspondence before the matter was settled. The misconception of course was in the use and meaning of the word rotation.

86. The successive positions of the instantaneous axis in space will form a cone, fixed in space; and the successive positions of the instantaneous axis in the body will form a cone, fixed in the body, and the whole motion will be represented by the rolling of this cone upon the fixed cone.

These are sometimes called the fixed and moving axodes.

An important application of this idea is the discussion of the motion of a body under the action of no force, in Poinsot's *Nouvelle Théorie de la rotation des corps solides.*

87. It may be useful to indicate a different method of dealing with the ideas of the two preceding articles.

Consider the body to be rigidly attached to a sphere, the centre of which is at the fixed point; then the motion of the sphere will determine the motion of the body.

If P and Q are two points on the surface of the sphere, the motions of P and Q determine the motion of the sphere.

Through P and Q draw great circles perpendicular respectively, to the directions of motion of P and Q; these great circles intersect in a point C, which has no motion, and OC is therefore the instantaneous axis.

And, exactly as in Art. (60), the locus of C on the surface of the moving sphere will be a spherical curve rolling on the arc of a fixed spherical curve, the locus of C in space, thus constituting the fixed and moving spherical centrodes.

88. *A circle rolls on the arc of a fixed circle, the plane of the rolling circle being inclined at a given angle to the plane of the fixed circle; it is required to find the position of the instantaneous axis.*

Take a and c as the radii of the fixed and rolling circles, and α as the inclination of their planes to each other.

O and C being the centres of the circles, the normals to their planes through O and C meet in a fixed point E, fig. 60, and the motion is completely represented by the rolling of the right circular cone vertex E, and vertical angle PEQ, upon the fixed right circular cone, vertex E and vertical angle double the angle OEP.

The line EP is therefore the instantaneous axis.

The angle OPQ being α, let the angle $OPE = \phi$; then angle

$$PEC = \frac{\pi}{2} - (\alpha - \phi) = \frac{\pi}{2} - \alpha + \phi.$$

Projecting EC and OCP upon the plane of the fixed circle,

$$EC \sin \alpha = a + c \cos (\pi - \alpha) = a - c \cos \alpha.$$

$$\therefore \quad EP \sin (\alpha - \phi) \sin \alpha = a - c \cos \alpha,$$

and $$a \sin (\alpha - \phi) \sin \alpha = (a - c \cos \alpha) \cos \phi,$$

leading to $$\tan \phi = \frac{c - a \cos \alpha}{a \sin \alpha}.$$

If α is greater than $\frac{\pi}{2}$, ϕ is positive, but if α is less than $\frac{\pi}{2}$, and $a \cos \alpha > c$, ϕ is negative, and the instantaneous axis PE is beneath the plane of the fixed circle, results which are at once obvious from the figure.

The semi-vertical angles of the fixed and rolling cones are respectively $\frac{\pi}{2} - \phi$, and $\frac{\pi}{2} - (\alpha - \phi)$, and the tangents of these angles are respectively

$$\frac{a \sin \alpha}{c - a \cos \alpha}, \text{ and } \frac{c \sin \alpha}{a - c \cos \alpha}.$$

Again, if V is the velocity with which the point of contact moves round,

$$\text{the velocity of } C = V \frac{a - c \cos \alpha}{a}.$$

Therefore the angular velocity of the disc about the instantaneous axis EP

$$= V \frac{a - c \cos \alpha}{a} \div c \sin (\alpha - \phi)$$

$$= V \left\{\frac{1}{a^2} + \frac{1}{c^2} - \frac{2 \cos \alpha}{ac}\right\}^{\frac{1}{2}}.$$

In the general case of any plane curve rolling on another plane curve, this is the relation between the angular velocity, and the velocity of the point of contact, if a and c are the radii of curvature at the point of contact.

89. The rate of rotation is the angular velocity about the axis, or the *spin*, as it was called by the late Professor Clifford. It is measured by the rate of increase of the inclination of a fixed plane in the body, containing the axis, to a fixed plane in space, containing the axis.

If OA is the axis of a spin, it is considered to be positive when the motion is clockwise, looking in the direction AO, or counter-clockwise, looking in the direction OA.

Composition of Rotations.

90. *Parallelogram of angular velocities, or parallelogram of spins.*

If OA, OB are the axes of two spins, and if the lengths of OA and OB are proportional to the magnitudes of the spins, the resultant spin is represented in magnitude and direction by the diagonal OC of the parallelogram formed by OA and OB.

Taking ω and ω' as the magnitude of the spins, the velocity of any point P in the plane AOB, perpendicular to that plane,

$$=\omega PD+\omega' PE, \text{ fig. } 61,$$

PD and PE being the perpendiculars on OA and OB.

Now, if PD meets BC, or BC produced, in K,

$$OA\,.\,PD=OA\,(PK+KD)=BC\,.\,PK+OA\,.\,KD$$

$$=2\,(PBC+OBC),$$

and $$OB\,.\,PE=2POB.$$

$$\therefore\quad OA\,.\,PD+OB\,.\,PE=2\,.\,POC=OC\,.\,PF,$$

if PF is the perpendicular on OC.

Hence, if the ratio of Ω to ω is the same as that of OC to OA, the velocity of P

$$= \Omega . PF,$$

and therefore OC represents the resultant spin in direction and magnitude.

91. *Resultant of two spins about parallel axes.*

Suppose a rigid body to have, at any instant, two spins, ω and ω', about parallel axes through the points A and B.

Take any point P in the plane containing these axes, and let PAB be perpendicular to the axes. The velocity of P will be in the direction perpendicular to the plane, and its magnitude will be

$$\omega AP + \omega' BP.$$

This will vanish if P coincide with a point C between A and B, such that

$$\omega AC = \omega' BC,$$

and the velocity of P will be $(\omega + \omega')\, CP$.

It follows that the resultant spin is of magnitude $\omega + \omega'$ about the axis through C parallel to the original axes.

If the spins about A and B are in contrary directions, and of numerical magnitude ω and ω', the velocity of D will be

$$\omega . AP - \omega' . BP,$$

and this will vanish if P coincide with a point C in BA produced, such that

$$\omega . AC = \omega' . BC,$$

and the velocity of P will then be $(\omega - \omega')\, CP$.

If in this case $\omega = \omega'$, the velocity of any point in the plane will be

$$\omega . AB,$$

and therefore, equal and opposite spins about parallel axes are equivalent to, and may be represented by, a motion of translation.

92. *A sphere rolls on a plane so that its centre moves in a circle.*

Let P be the point of contact, O the centre of the circle, radius a, on which it moves, and C the centre of the sphere, of radius c.

Then, if the sphere have no spin about the vertical diameter, OP is the instantaneous axis, and therefore the motion is represented by the rolling of the cone, axis OC and semi-vertical angle COP, upon the plane.

The cone is the moving axode, and the plane is the fixed axode.

If Ω is the angular velocity of the point C about the normal to the plane through the point O, and if Ω' is the spin about the instantaneous axis, looking in the direction PO, it follows that Ω' is negative, and that its numerical value is $a\Omega/c$.

93. If the sphere have a constant spin ω about the vertical diameter, a state of things which exists under certain dynamical conditions, the instantaneous axis is the line EP, such that

$$OE : OP :: \omega : \Omega', \text{ fig. (62)},$$

and that the fixed axode is the right circular cone, axis EO, and vertical angle OEP, and that the moving axode is the circular cone, axis EC and vertical angle PEC, so that the small circle PQ is the circle of contact.

94. *A sphere rolls on a surface of revolution, so that its centre moves in a circle.*

If P is the point of contact, and if PO, the tangent to the meridian at the point P, meets the axis Oz of the surface in the point O, PO will be the instantaneous axis, provided that the sphere has no angular velocity about the normal at P, a state of things which is possible.

In this case the fixed axode will be the cone, axis Oz, and vertical angle twice POz, and the moving axode will be the cone, having its vertex at the point O, and enveloping the sphere.

If the sphere should have a constant velocity about the normal at P, which is dynamically possible, the instantaneous axis will be a straight line through P meeting Oz in a fixed point E. The fixed axode will then be the cone, axis Ez, and vertical angle twice PEz, and the moving axode will be the cone, having EC for its axis, and CEP for its semi-vertical angle.

Motions of Translation and Rotation combined.

95. If one point of a rigid body is fixed, it is clear that the body cannot have a motion of translation, but that it may have a spin about some axis through the point.

It follows therefore that any state of motion of a rigid body can be represented by a motion of translation, combined with a spin about some axis.

It will be shewn in the next article that this state of motion can always be transformed into a translation in some direction, combined with a spin about an axis in that direction, that is by a *spin on a screw*.

This screw is called the instantaneous screw.

The successive positions of its axis, in space and in the body, create two ruled surfaces, which are called respectively the fixed and moving axodes.

These axodes may be developable surfaces, or skew surfaces, or, according to Professor Cayley's nomenclature, *torses* or *scrolls*, and the motion of the body is completely represented by the rolling and sliding of the one axode on the other.

96. If we have given the state of the motion of one point of a body at any instant, and the rotation of the body about some axis through the point, it is clear that the motion of the body is completely determined.

With these data we can find the instantaneous screw.

Let u be the velocity of a point O of a body, and OA its direction of motion, and let the body have the spin ω about the axis OB.

Draw the straight line OE perpendicular to the plane AOB, fig. (63), and through the point E of the line draw EF parallel to OB.

Apply to the body two equal and opposite spins, of magnitude ω, about the axis EF.

The spins ω and $-\omega$ about OB and EF are equivalent to the translation ωr, if $OE = r$, in the direction perpendicular to the plane BOE.

The resultant translation will be in the direction EF, if

$$r\omega = u \sin \alpha,$$

α being the angle AOB, and if v is the magnitude of the motion of translation,

$$v = u \cos \alpha.$$

EF is therefore the axis of the instantaneous screw, and the motion is represented by the velocity v in the direction EF, and the spin ω about EF.

97. It will now be seen that when the motion is represented by a spin on a screw, the magnitude of the translation is the least possible.

In fact, if we apply two equal and opposite spins, each equal to the screw spin, about any axis parallel to the axis of the screw, we shall obtain a spin about this parallel axis and a translation perpendicular to it, which, combined with the original translation, will produce a translation of greater magnitude.

98. Other modes of representing the motion of a body may be adopted.

For instance, any state of motion can be represented by two spins about axes at right angles to each other, in an infinite number of ways. To prove this, take any point E in the axis of the instantaneous screw EF, fig. (64), and, in any plane through EF, draw two straight lines EA, EB at right angles to each other, and draw PEQ perpendicular to the plane AEB.

If v and ω represent the screw motion, and if the angle $FEA = \theta$, these are equivalent to the two screws,

$$v \cos \theta,\ \omega \cos \theta,\ \text{ and }\ v \sin \theta,\ \omega \sin \theta.$$

Through P and Q draw straight lines PC, QD parallel to EA and EB.

Apply two equal and opposite spins, $\omega \cos \theta$, to PC, and two equal and opposite spins, $\omega \sin \theta$, to QD.

If we take P and Q such that

$$QE \,.\, \omega \sin \theta = v \cos \theta,$$

and
$$PE \,.\, \omega \cos \theta = v \sin \theta,$$

all translations will disappear, and we shall be left with two spins, $\omega \cos \theta$ about PC, and $\omega \sin \theta$ about QD.

Conversely if we are given two spins, α and β, about axes at right angles, and at a given distance c from each other, we can determine the instantaneous screw.

For
$$\omega \cos \theta = \alpha, \text{ and } \omega \sin \theta = \beta;$$

$$\therefore \quad \omega = \sqrt{\alpha^2 + \beta^2}, \text{ and } \tan \theta = \frac{\beta}{\alpha}.$$

Also,
$$c = PE + QE = \frac{v}{\omega \sin \theta \cos \theta};$$

$$\therefore \quad v = c\omega \sin \theta \cos \theta = \frac{c\alpha\beta}{\sqrt{\alpha^2 + \beta^2}},$$

and
$$PE : QE :: \beta^2 : \alpha^2,$$

so that the screw is completely determined.

Pitch of a Screw.

99. In the case of a screw motion, when v is the velocity of translation, and ω the spin, if we take a quantity p such that

$$v = p\omega,$$

p is called the pitch of the screw.

If the motion continues uniform, p is the actual translation due to a twist through the unit of angular measure.

Composition of Screws.

100. *The axes of two screws intersect at right angles; it is required to find their resultant.*

Let v, ω and v', ω' represent the two screws; then if p and p' are the pitches,

$$v = p\omega \text{ and } v' = p'\omega'.$$

Ox and Oy being the axes of the screws, fig. (65), let OP be the axis of the resultant Ω of the two spins ω and ω'.

Also let OQ be the direction of the resultant V of the two velocities v and v'.

The motion is then reduced to the translation V in OQ and the spin Ω about OP.

Let the angle $POx = \theta$, and $QOx = \phi$.

In the line Oz, perpendicular to the plane xOy, take a point E at the distance z from O, and draw EF parallel to OP.

Apply to the body two equal and opposite spins about EF, each equal to Ω.

The motion then consists of the spin Ω about EF, the translation Ωz perpendicular to the plane EOP, and the translation V in OQ.

The resulting translation is in the direction EF, if

$$z\Omega = V \sin(\phi - \theta) = v' \cos\theta - v \sin\theta,$$

leading to $$z\Omega^2 = (p' - p)\,\omega\omega' \quad \ldots\ldots\ldots\ldots(1),$$

or $$z = (p' - p) \sin\theta \cos\theta.$$

This determines the position of the axis of the resultant screw, the spin of which is Ω, and the translation U, where

$$\Omega = \sqrt{\omega^2 + \omega'^2},$$

and $$U = V\cos(\phi - \theta) = v\cos\theta + v'\sin\theta$$
$$= p\omega\cos\theta + p'\omega'\sin\theta$$
$$= \Omega\,(p\cos^2\theta + p'\sin^2\theta),$$

so that if ϖ is the pitch of the resultant screw,

$$\varpi = p\cos^2\theta + p'\sin^2\theta.$$

If we measure θ from the line bisecting the angle between the axes of the screws, that is, if we write $\frac{\pi}{4} + \theta$ for θ we obtain

$$2z = (p' - p)\cos 2\theta,$$

and $$2U = \Omega\,\{p' + p + (p' - p)\sin 2\theta\}.$$

The Cylindroid.

101. If p and p' are given, and if ω and ω' are allowed to change, the angle θ will be variable, and the position of the instantaneous axis will change.

It will trace out a skew surface, conoidal, of which Oz is the axis, and, since the equations defining its position are

$$z = (p' - p)\sin\theta\cos\theta, \quad y = x\tan\theta,$$

it follows by the elimination of θ that

$$z\,(x^2 + y^2) = (p' - p)\,xy$$

is the equation of the conoidal surface.

This surface is called a cylindroid, in the nomenclature of Sir R. S. Ball (Ball's *Theory of Screws*).

Turning the axes of x and y through half a right angle, the equation takes the form

$$2z\,(x^2 + y^2) = (p' - p)\,(x^2 - y^2).$$

102. *The axes of two screws intersect at the angle* 2α; *it is required to find their resultant.*

Take Ox and Oy bisecting the angles 2α and $\pi - 2\alpha$, the screws p and p' having for their axes OA and OB respectively.

The translations are $(v'+v)\cos\alpha$, and $(v'-v)\sin\alpha$, and the spins about Ox and Oy are

$$(\omega'+\omega)\cos\alpha, \text{ and } (\omega'-\omega)\sin\alpha.$$

Taking Ω and V as the resultants and θ, ϕ, as the inclinations to Ox, we have

$$V^2 = v^2 + v'^2 + 2vv'\cos 2\alpha, \quad \Omega^2 = \omega^2 + \omega'^2 + 2\omega\omega'\cos 2\alpha.$$
$$V\cos\phi = (v'+v)\cos\alpha, \quad \Omega\cos\theta = (\omega'+\omega)\cos\alpha,$$
$$V\sin\phi = (v'-v)\sin\alpha, \quad \Omega\sin\theta = (\omega'-\omega)\sin\alpha.$$

As in Art. 100, if we take $z\Omega = V\sin(\phi-\theta)$,
we obtain $$z\Omega^2 = (p'-p)\,\omega\omega'\sin 2\alpha \qquad (1),$$
which becomes equation (1) of Art. 100, when AOB is a right angle.

The spin of the resultant screw is Ω, and, if U is the translation

$$U = V\cos(\phi-\theta),$$

so that

$$\Omega . U = (p'\omega'+p\omega)(\omega'+\omega)\cos^2\alpha + (p'\omega'-p\omega)(\omega'-\omega)\sin^2\alpha,$$
$$= p\omega^2 + p'\omega'^2 + (p'+p)\,\omega\omega'\cos 2\alpha.$$

Taking ϖ as the pitch of the resultant screw, and observing that

$$\omega'\sin 2\alpha = \Omega\sin(\alpha+\theta), \text{ and } \omega\sin 2\alpha = \Omega\sin(\alpha-\theta),$$

we obtain the equation,

$$\varpi\sin^2 2\alpha = p'\sin^2(\alpha+\theta) + p\sin^2(\alpha-\theta)$$
$$+ (p'+p)(\sin^2\alpha - \sin^2\theta)\cos 2\alpha \qquad (2).$$

Again, we obtain, from (1),

$$z\sin 2\alpha = (p'-p)(\sin^2\alpha - \sin^2\theta),$$

and if we put $y = x\tan\theta$, and eliminate θ, we find that

$$z(x^2+y^2)\sin 2\alpha = (p'-p)(x^2\sin^2\alpha - y^2\cos^2\alpha),$$

is the Cartesian equation of the surface traced out by the axis of the resultant screw when p and p' are fixed, and ω, ω' are variable.

103. *The shortest distance between the axes of two screws is* 2c, *and* 2α *is the inclination to each other of the axes; it is required to find the resultant screw.*

Take the middle point of the shortest distance as origin, and as axes of x and y, take the straight lines bisecting the angles 2α and $\pi - 2\alpha$.

Let AC be the axis of the screw p, and take for its equations

$$y = -x \tan \alpha, \quad z = -c.$$

Then, if BD is the axis of the screw p', its equations are

$$y = x \tan \alpha, \quad z = c.$$

Let OK and OL be the projections on the plane xy of the axes AC and BD.

Applying equal and opposite spins about OK and OL, the screw p is equivalent to the translation v in OK, the spin ω about OK and the translation $c\omega$ perpendicular to OK, in the direction figured. Fig. (66).

Similarly the screw p' is equivalent to the translations v' in OL, the spin ω' about OL, and the translation $c\omega'$ in the direction perpendicular to OL.

The motion is therefore equivalent to the translations, parallel to Ox and Oy,

$$(v'+v)\cos\alpha - (\omega'+\omega)c\sin\alpha,\ (v'-v)\sin\alpha + (\omega'-\omega)c\cos\alpha \ldots (1),$$

and the spins about Ox and Oy,

$$(\omega'+\omega)\cos\alpha, \quad (\omega'-\omega)\sin\alpha \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

As before take Ω and V as the resultants, θ and ϕ as the inclinations to Ox, so that the above quantities are equal to

$$V\cos\phi, \quad V\sin\phi, \quad \Omega\cos\theta, \quad \Omega\sin\theta.$$

If we take z such that

$$z\Omega = V \sin(\phi - \theta) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3),$$

we obtain

$$z\Omega^2 = (p' - p)\,\omega\omega' \sin 2\alpha + (\omega'^2 - \omega^2)\,c,$$

which determines the position of the resultant screw axis.

If $c = 0$, we obtain the result of Art. 102.

The spin of the resultant screw is Ω, and if U is the translation,

$$U = V \cos(\phi - \theta) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

leading to

$$\Omega U = p'\omega'^2 + p\omega^2 + (p' + p)\,\omega\omega' \cos 2\alpha - 2c\omega\omega' \sin 2\alpha.$$

If we put $y = x \tan\theta$, we obtain, from (3),

$$z(x^2 + y^2) \sin 2\alpha = (p' - p)(x^2 \sin^2\alpha - y^2 \cos^2\alpha) + 2cxy \sin 2\alpha.$$

104. Conversely, any screw can be decomposed into two screws having their axes in any two planes parallel to the axis of the given screw, and having these axes inclined to each other at any given angle.

These conditions fix the values of α, c, and z, and, if we take U and Ω as the elements of the given screw, we can assume θ at pleasure, so that V and ϕ will be determined by the equations (3) and (4), and then the equivalences of $V \cos\phi$, $V \sin\phi$, $\Omega \cos\theta$, $\Omega \sin\theta$ to the expressions (1) and (2) will determine the elements of the two screws.

105. *To prove that a cylindroid is completely determined if two screws are given.*

Adopting the notation of Art. 103, let axes of ξ and η, parallel to the plane xy, meet in a point on the axis of z at a depth ζ below the plane of xy, and let ψ be the inclination of the axis of x to the axis of ξ, so that $\psi + \alpha$ and $\psi - \alpha$ are the inclinations to the axis of ξ of the axes of the screws p' and p.

Also let ϖ and ϖ' be the pitches of the ξ and η screws.

Then if the p and p' screws are on the cylindroid defined by ϖ and ϖ', we have from Art. 100,

$$p' = \varpi \cos^2 (\psi + \alpha) + \varpi' \sin^2 (\psi + \alpha) \quad\ldots\ldots\ldots\ldots(1),$$

$$p = \varpi \cos^2 (\psi - \alpha) + \varpi' \sin^2 (\psi - \alpha) \quad\ldots\ldots\ldots\ldots(2),$$

$$\zeta + c = (\varpi' - \varpi) \sin (\psi + \alpha) \cos (\psi + \alpha) \quad\ldots\ldots\ldots(3),$$

$$\zeta - c = (\varpi' - \varpi) \sin (\psi - \alpha) \cos (\psi - \alpha) \quad\ldots\ldots\ldots(4).$$

From (1) and (2) we obtain

$$p' - p = (\varpi' - \varpi) \sin 2\psi \sin 2\alpha,$$

and from (3) and (4),

$$2c = (\varpi' - \varpi) \cos 2\psi \sin 2\alpha \quad\ldots\ldots\ldots\ldots\ldots\ldots(5);$$

and $\therefore$ $2c \tan 2\psi = p - p'$, which determines ψ.

Again $$p' + p = \varpi' + \varpi - (\varpi' - \varpi) \cos 2\psi \cos 2\alpha$$
$$= \varpi' + \varpi - 2c \cot 2\alpha \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

and, adding together (3) and (4),

$$2\zeta = (\varpi' - \varpi) \sin 2\psi \cos 2\alpha$$

or $$2\zeta = (p' - p) \cot 2\alpha, \text{ which determines } \zeta.$$

Lastly ϖ and ϖ' are given by the equations (5) and (6), and the cylindroid is therefore completely determined.

It is obvious that the resultant of the screws p and p' is on the cylindroid given by ϖ and ϖ'.

For each screw is decomposable into two having ξ and η for axes, and the two pairs of ξ and η components are each equivalent to one ξ component and one η component, and therefore to a screw on the cylindroid.

It will be seen also that the spins of the three screws follow the parallelogrammic law and are therefore in the ratio of the sines of the angles between the axes of the screws.

106. *The equations of the axis of the instantaneous screw.*

The motion of a rigid body being completely represented by the motion of a point of the body, and a rotation about some axis through that point, let u, v, w be the component velocities of the point O', and $\omega_1, \omega_2, \omega_3$ the component angular velocities about axes through O'.

If x', y', z' are coordinates of a point P of the body referred to O', its velocities relative to O' are

$$z'\omega_2 - y'\omega_3, \quad x'\omega_3 - z'\omega_1, \quad y'\omega_1 - x'\omega_2,$$

and the actual velocities of P are obtained by adding u, v, w to these expressions.

The point P will be a point in the screw axis if the direction of motion of P is coincident with the direction of the axis of resultant angular velocity, that is, if

$$\frac{u + z'\omega_2 - y'\omega_3}{\omega_1} = \frac{v + x'\omega_3 - z'\omega_1}{\omega_2} = \frac{w + y'\omega_1 - x'\omega_2}{\omega_3}.$$

These then are the equations of the screw axis referred to O', and, knowing the position of the axes through O' relative to axes fixed in space, we can determine the position, relative to those fixed axes, of the screw axis.

107. *A sphere rolls between two parallel surfaces of revolution which are rotating about their common axis with different angular velocities; it is required to determine the motion, and the axodes.*

If P and Q are the points of the sphere in contact with the surfaces at the distances r and r' from the axis, the velocities of P and Q are the same as those of the points in the surface with which they are in contact, and are therefore, fig. (67),

$$\omega r \text{ and } \omega' r'$$

in the direction perpendicular to the plane CGE.

Hence if V is the velocity of C,

$$V = \tfrac{1}{2}(\omega r + \omega' r'),$$

in the same direction, and it follows that C moves in the circle centre N, with the velocity V.

If the system be started from a state of repose, the only angular velocity of the sphere about C will be about the axis CA, and if Ω be this angular velocity,

$$\Omega = \frac{\omega' r' - \omega r}{2c}.$$

If we take the point F such that

$$V = \Omega \, . \, CF,$$

the instantaneous axis is the line FE parallel to CA, and meeting the axis of the surfaces in a fixed point.

Hence the fixed axode is the right circular cone, vertex E, axis EG, and vertical angle $2FEG$, and the moving axode is the cone, vertex E, axis EC and vertical angle $2FEC$.

108. *A sphere rolls between two parallel planes, which are rotating, with equal angular velocities in the same direction, about fixed axes perpendicular to the planes; to determine the motion and the axodes.*

Let the plane through C, the centre of the sphere, parallel to the given planes, intersect the two axes of rotation in A and B, and if P, Q are the points of contact, take P above this plane, and Q beneath it.

Take c for the radius of the sphere and let

$$AE = EB = a.$$

Starting the system by suddenly setting the planes in motion, the velocities of P parallel to x and y, i.e. perpendicular and parallel to AB, are $\omega \, . \, AN$ and ωCN; and the velocities of Q are ωBN and ωCN, N being the projection of C upon AB.

Therefore the velocities of C parallel to x and y are

$$\omega EN \text{ and } \omega . CN,$$

that is, the point C moves uniformly in the circle, centre E and radius EC (b say).

Again there is no angular velocity about Cx, because the velocities of P and Q in the direction Cy are equal; but the angular velocity about Cy

$$=\frac{\omega . AN - \omega BN}{2c} = -\omega\frac{a}{c} = \Omega \text{ say}.$$

The motion is now represented by the translation ωb in CK perpendicular to EC, and the rotation Ω about Cy.

The screw axis is parallel to Cy and is at the height above C given by the equation

$$z\Omega + \omega b \cos\psi = 0, \text{ or } \omega\frac{a}{c}z = \omega b\cos\psi,$$

the angle KCx being ψ, so that

$$z = \frac{c}{a}. EN.$$

The spin about the screw axis is $-\omega\frac{a}{c}$, and the translation is $\omega b \sin\psi$.

The axodes are clearly cylindrical surfaces, and if $CN = x$, we have $x = b\sin\psi$,

$$\therefore \frac{x^2}{b^2} + \frac{a^2z^2}{b^2c^2} = 1,$$

is the equation of the fixed axode.

To find the moving axode, observe that while EC turns through the angle ψ, a vertical diameter of the sphere turns through the angle $\frac{a}{c}\psi$, so that, taking ρ and ϵ as polar coordinates of the screw axis, referred to C and this moving diameter, we have

$$\rho = z = \frac{bc}{a}\cos\psi, \text{ and } \epsilon = \frac{a}{c}\psi;$$

$$\therefore \rho = \frac{bc}{a}\cos\frac{c\epsilon}{a},$$

is the polar equation of the moving axode. This moving axode rolls on the elliptic cylinder which is the fixed axode, and slips backwards and forwards with the variable velocity $\omega b \sin \psi$.

109. We have assumed in the preceding discussion that the system was originally at rest, and the planes set in motion, so that no rotation would be produced about the diameter perpendicular to the planes. If originally the sphere was rotating about this diameter, the angular velocity will remain unchanged, as we know from dynamical considerations.

The effect would be that the axodes would retain the same general forms, but that the axes of these cylinders would not be parallel to the planes.

110. It may be useful to obtain some of the results of Art. 108 in a different manner.

Taking Cx and Cy as axes, let the motion be represented by the velocities u, v, of C, and the rotations ω_1, ω_2 of the sphere about Cx and Cy.

We have to express the fact that the velocities of the points P and Q of the sphere are the same as those of the points of the planes with which they are in contact.

If $AC' = r$, $BC = r'$, $CAN = \theta$, $CBN = \phi$,

the conditions for the point P give the equations,

$$u + c\omega_2 = r\omega \cos \theta, \quad v - c\omega_1 = r\omega \sin \theta,$$

and, for Q,

$$u - c\omega_2 = r'\omega \cos \phi, \quad v + c\omega_1 = r'\omega \sin \phi;$$

we have also,

$$r \sin \theta = r' \sin \phi, \text{ and } r' \cos \phi - r \cos \theta = 2a.$$

Hence we obtain,

$$2u = \omega (r' \cos \phi + r \cos \theta), \quad 2v = \omega (r' \sin \phi + r \sin \theta),$$

$$c\omega_2 = -a\omega, \quad \omega_1 = 0;$$

$$\therefore \sqrt{u^2 + v^2} = \tfrac{1}{2}\omega \sqrt{r^2 + r'^2 - 2rr' \cos(\theta - \phi)} = \omega b,$$

and
$$\frac{v}{u} = \frac{2CN}{BN + AN} = \frac{CN}{EN} = \tan \psi,$$

shewing that the motion of C is in the circle of radius b, and that the only rotation is about Cy.

111. If in Art. 108 the rotation about the axis through B is in the opposite direction to that of the rotation about the axis through A, the effect is that the centre moves uniformly in the straight line CN, and that the sphere rotates with an angular velocity proportional to the length of EC about the axis which is inclined to Cx at the same angle as CK.

The fixed axode is a skew surface of the form,

$$bz(x^2 + y^2) + acxy = 0,$$

a conoidal surface, and it will be seen that the moving axode is also a skew surface.

MISCELLANEOUS EXAMPLES.

1. A smooth rigid wire bent into a curve turns round a fixed point in its own plane, and pushes a particle before it in a straight line. Find the form of the curve and shew that if it move with uniform angular velocity the particle moves with uniform velocity.

2. Q, C, P are fixed points in a rod. Q describes a circle whose centre is O and radius a, C describes a straight line passing through O; shew that generally P describes an egg-shaped oval, whose area is $\frac{\pi a^2 c}{b}$, and that the radii of curvature of its ends are

$$\frac{ac^2}{a(b+c) \pm b^2},$$

where $$QC = b,\ CP = c.$$

3. If a catenary rolls on a straight line, the envelope roulettes are involutes of parabolas.

4. An ellipse rolls on a fixed horizontal straight line (the axis of x). Shew that the locus of the highest point of the ellipse is given by the equation,

$$\frac{dx}{dy} = \frac{y^4 - 8a^2b^2}{y^2\sqrt{\{(4a^2 - y^2)(y^2 - 4b^2)\}}}.$$

5. An ellipse rolls on a straight line; prove that the difference between the lengths of the radii of curvature at corresponding points of the paths traced out by the foci is constant.

6. A cycloid rolls on an equal cycloid, corresponding points being in contact; shew that the locus of the centre of curvature of the rolling curve at the point of contact is a trochoid whose generating circle is equal to that of either cycloid.

7. Prove that the intrinsic equation of the envelope of the directrix of a catenary of parameter c rolling on a circle of radius c will be found by eliminating α between the equations

$$\frac{s}{c} = \frac{1}{2} \tan\alpha \sec\alpha + \frac{5}{4} \log \frac{1+\sin\alpha}{1-\sin\alpha},$$

and $$\phi = \alpha + \tan\alpha.$$

8. The cardioid $r = a(1-\cos\theta)$, rolls on a straight line; prove that the intrinsic equation of the roulette of the cusp is

$$2s = 3a(2\phi - \sin 2\phi),$$

measuring from the point of contact of the cusp.

Prove also that its Cartesian equation is

$$\frac{4a-x}{2a} = \left\{2 + \left(\frac{y}{2a}\right)^{\frac{2}{3}}\right\}\sqrt{1-\left(\frac{y}{2a}\right)^{\frac{2}{3}}},$$

that its area is $\frac{15}{4}\pi a^2$, and that the radius of curvature of the roulette of the cusp is three times its distance from the point of contact.

9. Shew that the problem for the determination of the caustic of a curve for rays proceeding from a point is the same as that of finding the evolute of the roulette traced out by the point corresponding to the given point, when an exactly equal curve is rolled upon the given curve, corresponding points being in contact.

Examine in particular what this becomes in the case of (i) rays proceeding from the focus of a parabola, (ii) rays proceeding from a point on a circle.

10. Prove that the curve on which an ellipse must roll in order that its centre may move in a straight line is given by the equation $y/a = \operatorname{dn} x/b$, the modulus being the eccentricity of the ellipse.

11. The right angle BAC slides so that AB, AC touch respectively two fixed circles; if c is the distance between the centres of these circles, prove that the fixed and moving centrodes are circles of diameters c and $2c$.

If r and r' are the radii of the two circles, and if t and t' are the distances from A to the points of contact, prove that the radius of curvature of the path of the point A is equal to

$$\frac{(t^2+t'^2)^{\frac{3}{2}}}{2(t^2+t'^2)-rt'-r't}.$$

Prove also that, if d is the distance between the points of contact, and c the distance between the centres of the circles, this expression is equal to

$$\frac{2d^3}{3d^2+c^2-r^2-r'^2}.$$

12. If a helix rolls on a straight line which it always touches, while its axis moves in a plane, any point of the helix traces a cycloid.

13. Two cylinders of different radii are placed on a table with their axes parallel. A board is placed upon them and drawn along in a direction perpendicular to the axes of the cylinders. If there be no slipping prove that the spaces passed over by the centres of the two cylinders are the same, and that each is equal to $\frac{1}{2}\frac{s}{\cos\alpha}$, where s = space traversed by the board, and $2x$ = angle between board and table.

14. Two equal circular discs of radius c with their planes parallel are fastened at their centres to a bar, the discs being inclined to the bar at the angle α. The two wheels thus formed being rolled along a plane, prove that the intrinsic equation of the track of either wheel upon the plane is

$$\sin\frac{s}{c}=\frac{\sin\phi}{\cos\alpha}.$$

15. The translations of two points of a rigid body are given in direction and magnitude, and there is no spin about the line joining them; find the screw axis.

16. A sphere rolls between two co-axial cylinders, which are rotating about their common axis; while one of them is sliding along its axis; prove that the path of the centre is a helix, and that the fixed axode is the surface generated by the tangent to a helix.

17. Find the fixed and moving axodes in the case of the steady motion of a top.

18. A sphere rolls between two concentric spheres, which are rotating about fixed diameters; determine the motion of its centre.

19. The vertex of the tangent cone to a fixed sphere, centre C, moves round the circumference of a fixed circle, centre O, in the plane perpendicular to OC, the cone having no rotation about its axis; find the axodes.

20. A sphere rolls between two parallel planes, which have different spins about fixed axes perpendicular to their planes; determine the motion of its centre.

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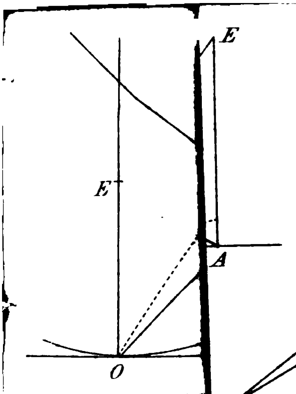
Plate 4.



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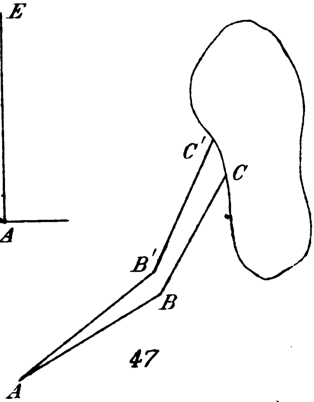

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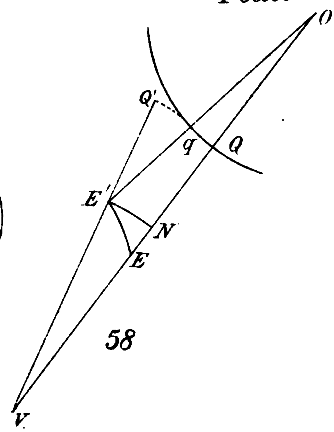

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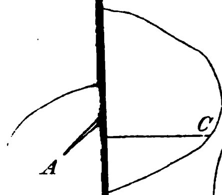

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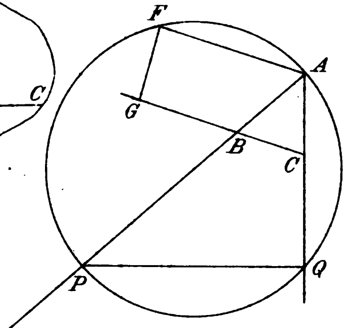

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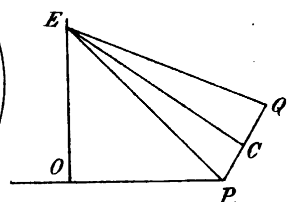

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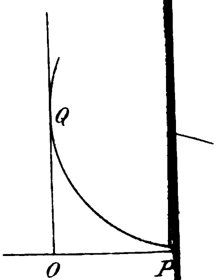

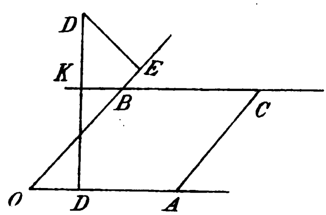

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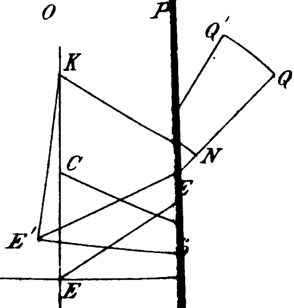

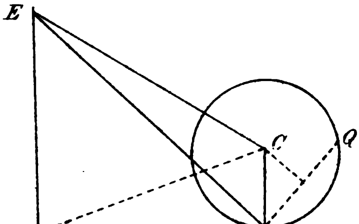

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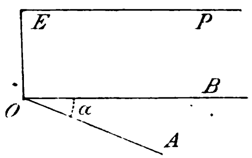

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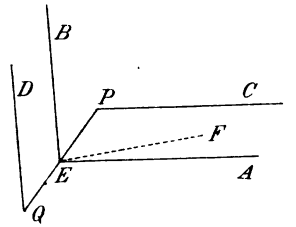

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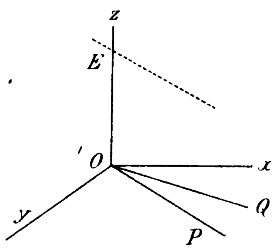

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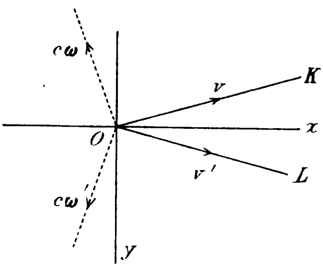

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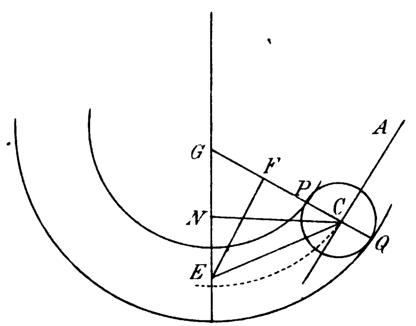

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