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ON

RETAINING WALLS

AND

MASONRY DAMS.

BY

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NEW YORK:

JOHN WILEY & SONS,

53 EAST TENTH STREET.

1892.

50550

ROBERT DRUMMOND,
Electrotyper,
444 and 446 Pearl St.,
New York.

FERRIS BROS.,
Printers,
326 Pearl Street,
New York.

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SLOPES, WALLS AND DAMS.

CHAPTER I.

EARTHWORK SLOPES.

ARTICLE 1. EQUILIBRIUM OF LOOSE EARTH.

Earthwork slopes are the surfaces formed when excavations, embankments, terraces, mounds, and other constructions are made in or with the natural earth. The earth is to be regarded in discussion as homogeneous and inelastic, and as consisting of particles more or less united by cohesion between which friction is generated whenever exterior forces tend to effect a separation. As some kinds of earth when dry are destitute of cohesion, these will first be considered under the term "loose earth."

The friction of earth upon earth will be taken to be governed by the same approximate laws as for other materials, namely: first, the force of friction between two surfaces is directly proportional to the normal pressure; second, it varies with the nature of the material; and third, it is independent of the area of contact. These laws may be expressed by the equation

$$F = fN, \qquad (1)$$

in which N is the normal pressure, F the force of friction perpendicular to N, and f is a quantity called the coefficient of friction which varies with the kind of material. As F and N are both in pounds, f is an abstract number; its value for earth ranges from about 0.5 to 1.0.

If a mass of earth be thoroughly loosened so as to destroy all cohesion between its particles, and then be poured vertically upon the point D in the horizontal plane BC, it will form a cone BAC, all of whose elements AB, AC, etc., make equal

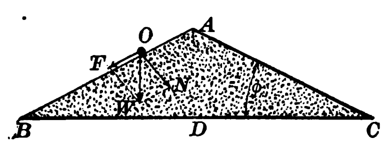


FIG. 1.

angles with the horizontal. This angle ABC is called the "angle of repose," or sometimes "the angle of natural slope," and it is found by experiment that each kind of earth has its own constant angle. The particles of earth on such a slope are held in equilibrium by the forces of gravity and friction. Let ϕ be the angle of repose ABD, and f the coefficient of friction. In the figure draw W vertically to represent the weight of a particle, and let N and F be its components normal and parallel to the slope. Now since motion is about to begin,

$$F = fN.$$

Also since the angle between N and W is equal to the angle of repose ϕ, the right-angled triangle NOW gives

$$F = N \tan \phi.$$

Therefore results the important relation

$$f = \tan \phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

that is, the coefficient of friction of earth is equal to the tangent of the angle of repose. It is hence easy to determine f when ϕ has been found by experiment.

In building an embankment of loose earth it is necessary that its slope, or angle of inclination to the horizontal, should not be greater than the angle of repose. When making an excavation it is often possible, on account of the cohesion of the earth, to have its slope at first greater than the angle of repose, but as the cohesion disappears under atmospheric influences the particles roll down and its slope finally becomes equal to ϕ.

The following table gives rough average values of the angles of repose and coefficients of friction of different kinds of earth. In the fourth column the inclination or slope is expressed in the manner customary among engineers by the ratio of its horizontal to its vertical projection. In the last column average values of the weight of the material are given.

Kind of Earth.	Angle of Repose. ϕ	Coefficient of Friction. f	Inclination. $\cot \phi$	Weight. Kilos per cu. met.	Weight. Pounds per cu. ft.
Gravel, round	30°	0.58	1.7 to 1	1600	100
Gravel, sharp	40	0.84	1.2 to 1	1700	110
Sand, dry	35	0.70	1.4 to 1	1600	100
Sand, moist	40	0.84	1.2 to 1	1700	110
Sand, very wet	30	0.58	1.7 to 1	2000	125
Earth, dry	40	0.84	1.2 to 1	1440	90
Earth, moist	45	1.00	1 to 1	1520	95
Earth, very wet	32	0.62	1.6 to 1	1840	115

It will be noticed that the natural slope and specific gravity of earth undergo quite wide variations as its degree of moisture varies. In collecting data for the discussion of particular cases it is hence necessary to determine limits as well as average values.

Problem 1. A bank of loose earth is 16 feet high, and its width, measured on the slope, is 28 feet. Compute the coefficient of friction and the angle of repose.

ARTICLE 2. THE COHESION OF EARTH.

Cohesion is a force uniting particles of matter together. If, for instance, two surfaces have been for some time in contact, they become to a certain extent glued or fastened together so that any attempt to separate them is met by a resistance. Friction only resists the separation of surfaces when motion is attempted which produces sliding, but cohesion resists their separation whether the motion be attempted parallel or perpendicular to the plane of contact. Particles of rock are held together by strong cohesive forces, while particles of dry sand have little, if any, cohesion.

By experiment the following are found to be the laws of cohesion: first, the force of cohesion between two surfaces is directly proportional to the area of contact; second, it depends upon the nature of the surfaces; and third, it is independent of the normal pressure. These laws may be expressed by the equation

$$C = cA, \quad \ldots \ldots \ldots \quad (3)$$

in which C is the resisting force of cohesion between two surfaces, A the area of contact, and c a quantity called the coefficient of cohesion depending upon the nature of the material.

The value of c for homogeneous earth may be found as follows: Dig in the ground several trenches of considerable length compared with their width, and of different depths. After a few days it will be observed that all those over a certain depth have caved along some plane such as BM in Figure 2. Let H be the value of this certain depth. Let w be the weight of a cubic unit of earth, and ϕ its angle of repose when devoid of cohesion. Then the coefficient of cohesion of the earth may be computed from the expression

$$c = \frac{Hw(1 - \sin\phi)}{4 \cos\phi}.$$

This formula will now be demonstrated.

Let the plane *BM* in the figure make an angle x with the horizontal. The prism *BAM*, whose length perpendicular to the drawing will be taken as unity, tends to slide down the

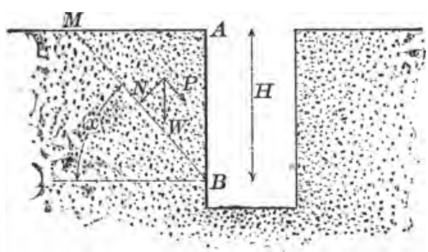


FIG. 2.

plane. Let *W* be the weight of this prism, and *P* and *N* its components parallel and normal to *BM*. *P* tends to cause motion down the plane, and this is resisted by the combined forces of friction and cohesion, acting in the plane. The force of friction is fN, and that of cohesion is cl, if l be the length, or area, of *BM*. At the moment of rupture

$$P = fN + cl,$$

from which the value of c is

$$c = \frac{P - fN}{l}. \qquad (4)$$

In order to determine the angle x, consider that the equations just written are for the plane of rupture, and that for any other plane P is less than $fN + cl$, or $\frac{P - fN}{l}$ is less than c. The value of x for the plane of rupture is hence that which renders

$$\frac{P - fN}{l} = \text{a maximum.} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Equation (5) will determine the value of x, and then c will be found from (4).

To do this insert first for P its value $W \sin x$, for N its value $W \cos x$, and for f its value $\tan \phi$. Then (5) becomes

$$\frac{W}{l}(\sin x - \tan \phi \cos x) = \text{a maximum.}$$

This may be written

$$\frac{W \sin (x - \phi)}{l \cos \phi} = \text{a maximum.}$$

Next express W in terms of H and x, and the weight of a unit of volume of earth w. Thus

$$W = \tfrac{1}{2} Hlw \sin (90° - x),$$

and then (5) becomes

$$\frac{Hw \sin (90° - x) \sin (x - \phi)}{2 \cos \phi} = \text{a maximum.}$$

This expression is a maximum when the two variable factors are equal; or when

$$90^\circ - x = x - \phi.$$

Thus the value of x for the plane of rupture is

$$x = 45^\circ + \tfrac{1}{2}\phi. \quad \ldots \ldots \quad (6)$$

Now to find c, insert in formula (4) the values of P, N, and f in terms of x, and it becomes

$$c = \frac{Hw \sin(90^\circ - x)\sin(x - \phi)}{2\cos\phi},$$

which by virtue of (6) reduces to

$$c = \frac{Hw \sin^2(45^\circ - \tfrac{1}{2}\phi)}{2\cos\phi}.$$

Since $\sin^2(45 - \tfrac{1}{2}\phi)$ equals $\tfrac{1}{2}(1 - \sin\phi)$, this value becomes

$$c = \frac{Hw(1 - \sin\phi)}{4\cos\phi}, \quad \ldots \ldots \quad (7)$$

which is the formula that was to be demonstrated.

From this formula the numerical value of c can be computed when H, w, and ϕ are known. For earth weighing 100 pounds per cubic foot and whose angle of natural slope is 30 degrees, the value of c becomes $14.4H$. If the vertical rupturing depth H is one foot, c is 14.4 pounds per square foot; but if H is ten feet, then c is 144 pounds per square foot.

Problem 2. A certain bank of earth, which has a natural slope when loose of 1.25 to 1, stands by virtue of its cohesion with a vertical face when $H = 3$ feet. If this bank fails, find the slope immediately after rupture.

ARTICLE 3. EQUILIBRIUM OF COHESIVE EARTH.

If the particles of earth be united by cohesion, a slope may exist steeper than the angle of repose. Let Figure 3 represent the practical case of an excavation ABC whose slope AB makes an angle θ with the horizontal greater than the

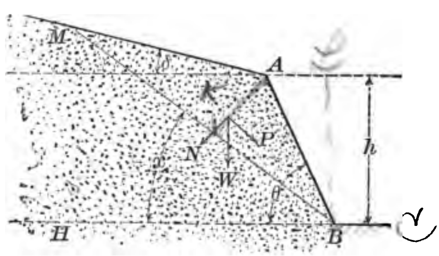


FIG. 3.

angle of repose ϕ. AM is the natural surface of the ground making with the horizontal an angle δ less than ϕ. It is required to determine the relation between the slope θ and the vertical depth h in order that rupture may just occur.

Let BM be the plane along which rupture occurs, and x its inclination to the horizontal. The weight of the prism BAM tends to urge it down the plane, and this is resisted by the forces of friction and cohesion acting in the plane. Let W be the weight of the prism for a length unity, and P and N its components parallel and normal to the plane. P is the force causing the downward sliding, fN is the resisting force of friction, and cl that of cohesion, if l be the area, or length, of BM. At the moment of rupture $P = fN + cl$, which may be written

$$\frac{P - fN}{l} = c. \quad \ldots \ldots \quad (8)$$

Now as x varies the forces, P and N vary; and for any other plane except that of rupture $\frac{P - fN}{l}$ is less than c. Hence the condition which will determine the value of x is

$$\frac{P - fN}{l} = \text{a maximum}. \quad \ldots \ldots \quad (9)$$

When x has been found from (9) its value is to be inserted in (8) and thus the relation between θ and h be established.

To do this insert in (9) for P and N their values $W \sin x$ and $W \cos x$, and for f its value $\tan \phi$. Then it takes the form

$$\frac{W \sin (x - \phi)}{l \cos \phi} = \text{a maximum}.$$

The value of W is the volume of the prism BAM, multiplied by the weight of a unit of volume w, or

$$W = \tfrac{1}{2} BA \, . \, BM \, . \, w \sin (\theta - x) = \tfrac{1}{2} hlw \, \frac{\sin (\theta - x)}{\sin \theta},$$

and hence the expression becomes

$$\frac{hw \sin(\theta - x) \sin(x - \phi)}{2 \cos\phi \sin\theta} = \text{a maximum.}$$

This is a maximum with respect to x when

$$\theta - x = x - \phi;$$

that is, the plane of rupture bisects the angle between the lines of natural slope and excavated slope, or

$$x = \tfrac{1}{2}(\theta + \phi). \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Now if (8) be expressed in terms of x, it becomes

$$hw \sin(\theta - x) \sin(x - \phi) = 2c \cos\phi \sin\theta,$$

and by virtue of (10) this reduces to

$$hw \sin^2 \tfrac{1}{2}(\theta - \phi) = 2c \cos\phi \sin\theta;$$

and substituting for $\sin^2 \frac{1}{2}(\theta - \phi)$ its value $\frac{1}{2}\big(1 - \cos(\theta - \phi)\big)$, and for c its value from (7), there is found

$$h\big(1 - \cos(\theta - \phi)\big) = H(1 - \sin\phi) \sin\theta, \quad . \quad . \quad (11)$$

and this is the equation of condition between h and θ.

This discussion shows that both the angle of rupture x and the relation between h and θ are independent of the slope δ made by the natural surface of the ground with the horizontal.

By the help of formula (11) the limiting height h may be found when θ, ϕ and H are given. For instance, let it be required to build a slope of 1 to 1, or $\theta = 45°$, and let the earth be such that $\phi = 30°$ and $H = 6$ feet. Then the depth at which rupture will occur is

$$h = 6 \frac{(1 - 0.5)0.707}{1 - 0.966} = 62 \text{ feet.}$$

For stability the depth must of course be less than 62 feet, and precautions be taken that the cohesion of the earth be not destroyed by the action of the weather.

Problem 3. Let a bank whose height is 30 feet and slope 45 degrees be of earth for which $\phi = 34°$ and $H = 3$ feet. How much higher can it be raised, keeping the same slope, before failure will occur?

ARTICLE 4. STABILITY OF SLOPES IN COHESIVE EARTH.

In practice it is desired to determine the slope of a bank so that it may be stable and permanent. To deduce an equation for this case consider again Figure 3, and let BM be any plane through the foot of the slope making an angle x with the horizontal. As x varies the forces P and N vary, and it is easy to see that the weakest plane is that for which the expression $\frac{P - fN}{l}$ is a maximum. As in Art. 3, the value of x rendering this a maximum is $x = \frac{1}{2}(\theta + \phi)$. Now it is required that rupture shall not occur along this plane, hence

$$P < fN + cl \quad \text{and} \quad P - fN < cl;$$

or if n be a number greater than unity, called the factor of security,

$$n(P - fN) = cl. \qquad (12)$$

Rupture can now occur only when the weight W becomes n times that of the prism of earth above the weakest plane. To adapt this equation to practical use it is only necessary to substitute for P and N their values in terms of x, and then to make x equal to $\frac{1}{2}(\theta + \phi)$. The substitution is performed exactly as before, and leads to the following result:

$$nh(1 - \cos(\theta - \phi)) = H(1 - \sin\phi)\sin\theta, \qquad (13)$$

which is the required equation of stability.

If n is unity, this of course reduces to the case of rupture as given by (11). The value to be assigned to the factor n can only be determined by observation and experiment on existing slopes. Probably about 2 or 3 will prove to be sufficient.

When θ is given, the value of h is derived at once directly from (13), thus:

$$h = \frac{H(1 - \sin\phi)\sin\theta}{n(1 - \cos(\theta - \phi))}. \qquad (14)$$

This shows that the height for security should be $\frac{1}{n}$ of that for rupture. Thus it was found in Article 3, if $H = 6$ feet, and $\phi = 30°$, that the limiting height for a slope of 45° would be 62 feet. Hence with a factor of security of 2 the height would be 31 feet, and with a factor of 3 the height would be 20 feet.

When h is given and θ is required, the formula for stability may be written in the form

$$\frac{1 - \cos(\theta - \phi)}{\sin\theta} = \frac{H(1 - \sin\phi)}{nh} = a.$$

The second member is here a known quantity and may be called a. By developing the numerator in the first member and then substituting for $\sin\theta$ and $\cos\theta$ their values in terms of $\tan\frac{1}{2}\theta$, a quadratic expression results whose solution gives

$$\tan\tfrac{1}{2}\theta = \frac{a + \sin\phi}{1 + \cos\phi} + \sqrt{-\tan^2\tfrac{1}{2}\phi + \left(\frac{a + \sin\phi}{1 + \cos\phi}\right)^2}. \quad (15)$$

This determines the slope θ for a factor of security n.

For example, let it be required to find the slope θ for a bank 25 feet high with a factor of security of 1.5, the value of ϕ being 30° and that of H being 6 feet. Here

$$a = \frac{5 \times 0.5}{1.5 \times 25} = 0.0667,$$

and then from the formula,

$$\tan \tfrac{1}{2}\theta = 0.304 + \sqrt{-0.0718 + 0.0922} = 0.447.$$

Hence $\frac{1}{2}\theta$ is about 24° and θ is about 48°, or a slope of 0.9 to 1. The slope when built must of course be protected from the action of the weather in order to preserve the cohesion of the earth.

The security of a bank may be investigated by measuring its height h and slope θ, and finding by experiment the angle of repose ϕ and vertical rupturing depth H. Then from (13) there is found

$$n = \frac{H(1 - \sin\phi)\sin\theta}{h(1 - \cos(\theta - \phi))}. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

For example, let it be required to find the factor n when $h = 30$ feet, $\theta = 45°$, $\phi = 34°$, and $H = 3$ feet. Substituting,

$$n = \frac{3 \times 0.441 \times 0.707}{30 \times 0.0184} = 1.7.$$

If such a slope had existed many years, and if the values of ϕ and H were the most unfavorable that could occur, it might be concluded that the factor of security deduced is sufficiently high; but if such a slope should be observed to fail, it would be necessary to conclude that the factor is too low.

Problem 4. A certain slope has $h = 25$ feet, $\phi = 30°$, and $H = 5$ feet. At what angle θ will rupture occur? What is its factor of security if θ be 48 degrees?

ARTICLE 5. CURVED SLOPES AND TERRACES.

The preceding articles clearly show that the angle of slope θ of a bank of cohesive earth increases as its vertical height h decreases, and, conversely, that as h becomes greater θ becomes smaller. It would hence appear that the upper part of a bank may be steeper than the lower part, and its liability to rupture be the same throughout. To determine the form of

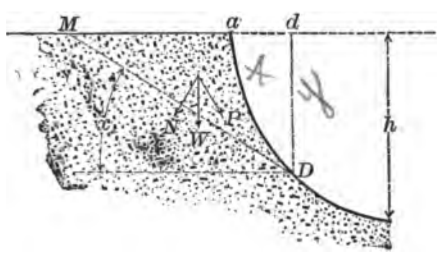


FIG. 4.

such a curve, let D in the figure be any point upon it whose ordinate Dd is y. Let DM be the weakest plane making an angle x with the horizontal. The prism of which aDM is a section, by virtue of its weight W, tends to slide down the plane. Let P and N be the components of W parallel and normal to the plane. If n be the factor of security, the condition, as in Art. 4, is

$$n(P - fN) - cl = 0.$$

By inserting for P, N and l their values in terms of x, this becomes

$$nW(1 - f \cot x) - cy(1 + \cot^2 x) = 0.$$

$$d \cot x = d\left(\frac{\cos x}{\sin x}\right) = \frac{\sin x(-\sin x) - \cos x \cos x}{\sin^2 x} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x = -(1 + \cot^2 x)$$

The value of W for a prism one unit in length is found from the difference of the areas dDM and dDa, or if A represent the surface dDa,

$$W = w(\tfrac{1}{2}y^2 \cot x - A).$$

By inserting this the equation of stability becomes

$$nw(\tfrac{1}{2}y^2 \cot x - A)(1 - f \cot x) - cy(1 + \cot^2 x) = 0. \quad (17)$$

i.e. negative i.e.

This expression equals zero for the weakest plane, but for any other plane its value is less than zero. Hence it must be a maximum with respect to x or $\cot x$, and its first derivative must vanish. Thus, also, Area A is constant, while x varies.

$$nw(\tfrac{1}{2}y^2 \cot x - A)(-f) + nw(1 - f \cot x)\tfrac{1}{2}(y^2) - 2cy \cot x = 0. \quad (18)$$

By eliminating $\cot x$ from (17) and (18), the following value of A is found:

$$A = \frac{y}{2nf^2w}\left(nfwy + 4c - 2\sqrt{2c(nfwy + 2c)(1 + f^2)}\right), \quad (19)$$

and this is the practical equation of the required curve, A being the area between the curve and any ordinate whose value is y.

For example, let it be required to construct a curve of equal stability in a bank of 40 feet height with a factor of security of 1.5, the earth having a natural slope of 31°, a vertical rupturing depth of 5 feet, and weighing 100 pounds per cubic foot. Here, from (2) and (7), there is first found

$$f = 0.6, \quad c = 71 \text{ pounds per square foot,}$$

and formula (19) becomes

$$A = \frac{y}{108}\left[90y + 284 - 2\sqrt{193\,(90y + 142)}\right].$$

From this are computed the following special values:

For $y = 10$ feet	$A = 27$ square feet;
For $y = 20$ feet,	$A = 159$ square feet;
For $y = 30$ feet,	$A = 421$ square feet;
For $y = 40$ feet,	$A = 809$ square feet.

These are the areas between the slope and the given ordinate, and may be practically regarded as consisting of trapezoids, as shown in Figure 5. The first area is that of the triangle *abB*, hence

$$\tfrac{1}{2} \cdot 10 \cdot ab = 27, \quad \text{or} \quad ab = 5.4 \text{ feet.}$$

The second area comprises the triangle *AbB* and the trapezoid *bBCc*, hence

$$27 + \frac{10 + 20}{2} \cdot bc = 159, \quad \text{or} \quad bc = 8.8 \text{ feet.}$$

In a similar manner $cd = 10.5$ feet and $de = 11.1$ feet, and the four points *B*, *C*, *D*, and *E* are thus located. In Figure 5 the portions of the slope are drawn as straight lines; it may be so built, or intermediate points of the curve be established by the eye.

It is not difficult to deduce from (19) the co-ordinate equation giving the relation between the abscissa and ordinate for every point of the curve, but it is of such a nature as to be of little practical use. In the manner just explained, as many points upon the curve may be located as required. It is seen from (19) that A is negative for small values of y, or theoreti-

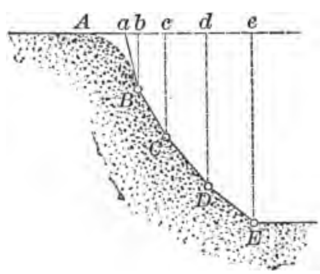


FIG. 5.

cally the curve overhangs the slope. Practically, of course, the equation should not be used for values of y less than H, and it will usually be found advisable and necessary that the upper part of the curve should be reversed in direction so as to form an ogee, as shown by the broken line AB in Figure 5.

When terraces are to be constructed, it is evident that the upper one may have the greatest slope and the lower one the least slope. Formula (19) may be used for this purpose, since the area A is not necessarily bounded by a curved line, but may be disposed in any form desired.

For example, take a bank 30 feet high in which it is desired to build three terraces, as in Figure 6, with a factor of safety of 1.5. The height of each terrace is 10 feet, and there are two steps *BC* and *DE*, each 4 feet wide. Let $w = 100$

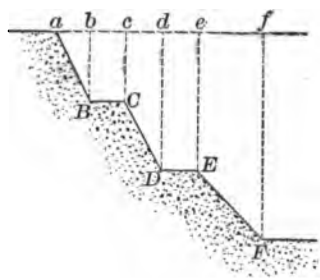


FIG. 6.

pounds, $\phi = 31°$, and $H = 5$ feet, as found by experiments. Then $f = 0.6$ and $c = 71$, and formula (19) becomes

$$A = \frac{y}{108}\left(284 + 90y - 2\sqrt{189(90y + 142)}\right).$$

From this, when $y = 10$, $A = 27$; when $y = 20$, $A = 159$; and when $y = 30$, $A = 421$. The abscissas are now found to be $ab = 5.4$, $cd = 6.1$, and $ef = 8.9$ feet. The three slopes are hence as follows:

For *aB*, $\cot\theta = \frac{5.4}{10}$ and $\theta = 61\frac{1}{2}°$;

For *CD*, $\cot\theta = \frac{6.1}{10}$ and $\theta = 58\frac{1}{2}°$;

For *EF*, $\cot\theta = \frac{8.9}{10}$ and $\theta = 48\frac{1}{2}°$.

To insure the permanency of these slopes they should be protected from the weather by sodding.

Problem 5. Design a terrace of four planes, the upper one being 6 feet in vertical height, the lowest 10 feet, and the others 8 feet; the steps to be 5 feet in width. The earth is such that cot $\phi = 1.5$ to 1, and $H = 3$ feet.

ARTICLE 6. PRACTICAL CONSIDERATIONS.

The preceding theory and formulas can be usually applied to the construction of embankments as well as to excavations, provided that care be taken to compact the earth to a proper degree of cohesion and the slopes be protected from the action of the elements. The height h is always given, and it is required to find the slope θ. Unless h be very large the weakest plane will intersect the roadway; but if not, the application of the formulas can only err on the side of safety. The load upon the roadway can be regarded as a mass of earth uniformly distributed over it and thus increasing the height h.

For instance, if $w = 100$ pounds per cubic foot, $\phi = 34°$ and $H = 4$ feet, let it be required to find the slope θ for an embankment 30 feet high. For security the weight of the locomotive should be taken high, say 6000 pounds per linear foot of track, or about 500 pounds per square foot of surface for a 12-foot roadbed, which would be equivalent in weight to a mass of earth about 4 feet high. Then the value of h to be used in formula (15) is 34 feet. If the factor of security be 2, the value of a is 0.0259, and

$$\tan \tfrac{1}{2}\theta = 0.320 + \sqrt{-0.0935 + 0.1024} = 0.414.$$

Hence $\frac{1}{2}\theta$ is about $22\frac{1}{2}$ degrees and θ is about 45°, or the slope is 1 to 1. The proposed embankment with this slope contains 47 cubic yards per linear foot, while with the natural slope of 34° it would contain 62 cubic yards per linear foot. A saving in cost of construction will hence result if the expense of protecting the slopes to preserve the cohesion be not too great.

The degree of moisture of earth exercises so great an influence upon its specific gravity and angle of repose that special pains should be taken to ascertain the values of those quantities which are the most unfavorable to stability. In general a high degree of moisture increases w and decreases ϕ. These causes alone would tend to increase the cohesion, but at such times H usually becomes so small that c is greatly diminished. The determination of H is awkward and there seem to be few recorded experiments concerning it. Care should be taken that the trench is long, or that transverse cuts be made at its ends so that lateral cohesion may not prevent rupture, and a considerable time should be allowed to elapse so that the cohesion may be subject to unfavorable weather.

The general conclusions of the above theory are valuable, but it should be applied with caution to particular cases, not only on account of the variability in the data but on account of our ignorance of the proper factor of security. Numerical computations, however, may often prove useful as guides in assisting the judgment. As shown above, a great saving in the cost of moving material will result if slopes be built in accordance with the theory, but evidently the cost of properly protecting the slopes will be increased. Should the latter cost prove to be the smaller, the theory will ultimately become of real practical value.

The preceding theory is not new, having long since been set forth in many French and German books, but the author is unaware to what extent it is practically used in those countries. The introduction of a factor of security is, however, believed to be novel in this connection, and by proper experiments for determining its value the practical application of the formulas here given may perhaps be rendered possible.

Problem 6. A railroad cut is to be made in material for which $w = 100$ pounds per cubic foot, $\phi = 32°$, $H = 5$ feet. If h is 40 feet and the roadbed 16 feet wide, find the quantity of material necessary to excavate when the slopes have a factor of security of 3. 940

CHAPTER II.

THE LATERAL PRESSURE OF EARTH.

ARTICLE 7. FUNDAMENTAL PRINCIPLES.

A retaining wall is a structure, usually nearly vertical, which sustains the lateral pressure of earth. In investigating the amount of this pressure it is generally regarded best to neglect the cohesion of the earth, and to consider it as loose (Article 1). This is done, partly because the effect of cohesion is difficult to estimate and partly because the results thus obtained are on the side of safety for the wall,—the entire investigation in fact being undertaken for the purpose of using the results in designing walls. The values given in Article 1 for the weight of earth and for the angles of repose will be used in this chapter, but it is again mentioned that they are subject to much variation, and that in practical problems the values most dangerous to stability should be selected.

The pressures against a retaining wall are least near the top and greatest near the base. The resultant of all these pressures is called the "resultant pressure," or simply the pressure, and is designated by the letter P. The determination of formulas for the values of P for different cases is the object of this chapter.

Let the resultant pressure P against the back of a wall be resolved into a component N acting normal, and a component F acting parallel, to the back of a wall. Let z be the angle between N and the direction of P; then

$$F = N \tan z.$$

Let f be the coefficient of friction between the earth and the wall, then for the case of incipient motion,

$$F = Nf.$$

Therefore, since f is the tangent of the angle of friction, the angle z cannot be larger than the angle of friction between the

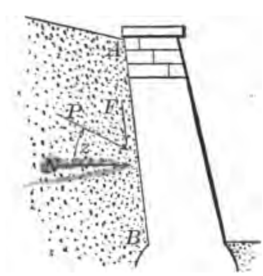


FIG. 7.

earth and the wall unless the earth is moving along the wall. Various views are held by authors regarding the direction of the pressure P, or the value of the angle z. Some take z as zero, or regard the thrust as normal to the wall; others take z

as equal to the angle of repose of the earth, ϕ; while a few take z as intermediate between these values.

In Article 8 the friction of the earth against the wall will be neglected, or the angle z will be taken as 0°. The value of the pressure determined under this supposition will be called the "normal pressure," and will be designated by P_1. It is not to be forgotten that the actual pressure against the back of a retaining wall cannot, like the pressure of water, be determined with certainty. The formulas to be deduced are such that, in general, they give limiting maximum values under the different conditions, and the hypothesis here adopted has the practical advantage of erring on the side of safety for the wall. In an unlimited mass of earth with horizontal surface, the pressure against any imaginary vertical plane must evidently be normal to that plane; now if a wall is to be designed to replace the earth on one side, the pressure against its back will also be normal. It would seem then, that the most satisfactory degree of stability of the earth will be secured by designing the wall under the assumption of normal pressure.

The views just expressed are, however, not accepted by some engineers who claim that the actual normal pressure is usually less than the values theoretically deduced for P_1, particularly for walls that have been observed to fail. In Article 9 there will hence be investigated formulas for the pressure supposing that it is inclined to the normal to the back of the wall at an angle ϕ; the value of the pressure thus derived will be called the "inclined pressure," and be designated by P_2.

Hence either P_1 or P_2 can be used in investigating the wall as the engineer thinks best.

Problem 7. Let the wall in Figure 7 be vertical, 12 feet in height, its thickness uniformly 2 feet, and its weight 3600 pounds. Let the point of application of P be 4 feet above the base. Compute the value of P to cause rotation, (a) when the angle β is 0°; (b) when the angle β is 30°.

ARTICLE 8. NORMAL PRESSURE AGAINST WALLS.

In Figure 8 is shown a wall which sustains the lateral pressure of a bank of earth. The back of the wall BA is inclined to the horizontal at the angle θ, and the surface of the earth AM

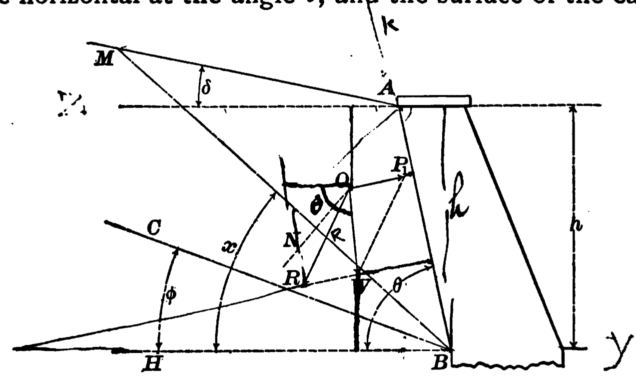


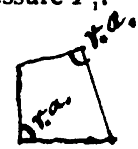
FIG. 8.

is inclined at the angle δ. The line BC represents the natural slope of the earth with the inclination ϕ. It is required to find the lateral pressure of the earth, supposing that its direction is normal to the back of the wall, or that, in Figure 7, the angle z is zero.

Draw BM making any angle x with the horizontal, and consider that the prism ABM in attempting to slide down the plane BM exerts a pressure upon AB. Let W be the weight of this prism, represented by the line OW, and let it be resolved into a component P_1 acting normal to the back of the wall, and a component R acting opposite to the direction of the reaction of the earth below BM. Let ON be normal to the plane BM, then the angle NOR will be equal to the angle of friction of earth upon earth, if the prism ABM is just on the point of sliding down BM; (for, as ON and NR are components of OR the triangle gives $NR = ON . \tan NOR$, but from the law of friction $NR = f . ON$; hence $f = \tan \phi = \tan NOR$, and accordingly $NOR = \phi$.)

Now in the triangle WOR the side OW represents the weight W, and the side WR the resultant normal pressure P_1. Hence

$$P_1 = W \frac{\sin WOR}{\sin WRO}.$$



* But the angle WOR is $x - \phi$ and the angle WRO is $\theta + \phi - x$. Let h be the vertical height of the wall, and w the weight of a cubic unit of earth; then the value of W for one unit in length of the wall is

$$W = \tfrac{1}{2} w . BA . BM . \sin ABM = \tfrac{1}{2} w h^2 \frac{\sin (\theta - \delta) \sin (\theta - x)}{\sin^2 \theta \sin (x - \delta)}.$$

*For $WON = x$ (sides perp.) and $NOR = \phi$. (as proved above) $\therefore WOR = WON - NOR = x - \phi$

The above value of P_1 then can be written

$$P_1 = \tfrac{1}{2}wh^2 \frac{\sin(\theta-\delta)\sin(\theta-x)\sin(x-\phi)}{\sin^2\theta\sin(x-\delta)\sin(\theta+\phi-x)}, \quad (20)$$

which expresses the normal pressure due to any prism whose plane BM makes an angle x with the horizontal. This expression becomes 0, both when $x = \theta$ and when $x = \phi$, and between those limits it has a maximum value which is to be taken as the pressure against the wall, since such can occur if the earth is about to slide down the corresponding plane.

In order to find the value of x which renders (20) a maximum it is best to write it in the form

$$P_1 = \frac{A(y-a)}{B(y-b)(y+c)}, \quad . \quad . \quad . \quad . \quad . \quad (21)$$

in which $y = \cot(\theta - x)$, $a = \cot(\theta - \phi)$, $b = \cot(\theta - \delta)$, $c = \cot\phi$, $A = \tfrac{1}{2}wh^2 \sin(\theta - \phi)$ and $B = \sin^2\theta \sin\phi$. Differentiating this with respect to y and putting the first derivative equal to zero, there is found

$$y = a + \sqrt{(a-b)(a+c)}, \quad . \quad . \quad . \quad . \quad (22)$$

and inserting this in (21) there results for the maximum

$$P_1 = \frac{A}{B(\sqrt{a+c} + \sqrt{a-b})^2}.$$

Thus P_1 is expressed in terms of the data w, h, θ, δ and ϕ, but to obtain a more convenient form it is well to replace the

$BA = \frac{h}{\sin\theta}$

cotangents by their equivalents in terms of sines. Then after reduction it becomes

only for: $\theta > \phi$ and $\delta < \phi$

$$P_1 = \frac{\frac{1}{2} w h^2 \sin^2 (\theta - \phi)}{\sin^2 \theta \left(1 + \sqrt{\dfrac{\sin \phi . \sin (\phi - \delta)}{\sin \theta \sin (\theta - \delta)}} \right)^2}, \quad . \quad . \quad . \quad (23)$$

which is the formula for the lateral normal pressure of a bank of earth against the back of a retaining wall.

This formula is valid for any value of θ greater than ϕ, and for any value of δ less than ϕ. By its discussion simpler formulas for special cases can be deduced.

The greatest value of δ will be that of the natural slope ϕ. For this case formula (23) becomes

$$P_1 = \tfrac{1}{2} w h^2 \frac{\sin^2 (\theta - \phi)}{\sin^2 \theta}, \quad . \quad . \quad . \quad . \quad (24)$$

which is the greatest normal thrust that can be caused by a sloping bank; if the wall be vertical $\theta = 90°$ and this reduces to the simple form $P_1 = \frac{1}{2} w h^2 \cos^2 \phi$.

The most common case is that where the surface *AM* is horizontal; for this $\delta = 0$ and (23) becomes

$$P_1 = \tfrac{1}{2} w h^2 \frac{\sin^2 \frac{1}{2}(\theta - \phi)}{\sin \theta . \sin^2 \frac{1}{2}(\theta + \delta)}, \quad . \quad . \quad . \quad . \quad (25)$$

which is the normal pressure of a level bank of earth against an inclined wall. If in this $\theta = 90°$, there results the formula for the pressure of a level bank against a vertical wall,

$$P_1 = \tfrac{1}{2}wh^2 \tan^2 (45 - \tfrac{1}{2}\phi),$$

which is the well-known expression first deduced by COULOMB in 1773.

Problem 8. Prove from (22) that, when $\delta = 0$, the plane BM bisects the angle between BA and BC. Prove it also by making $\delta = 0$ in (21), and then equating the first derivative to zero, thus deducing $x = \tfrac{1}{2}(\theta + \phi)$.

ARTICLE 9. INCLINED PRESSURE AGAINST WALLS.

In Figure 9 is shown a wall which sustains the lateral pressure of a bank of earth, the back of the wall being inclined to the horizontal at an angle θ, and its vertical height being h. The upper surface of the earth has an inclination δ to the horizontal, which is not greater than the natural slope ϕ. It is required to find the lateral pressure of the earth supposing that its direction makes an angle ϕ with the normal to the back of the wall.

Draw BM making any angle x with the horizontal, and consider that the prism BAM in attempting to slide down this plane is sustained by the reactions of the wall and of the earth below BM. Let OW represent the weight of this prism, and let it be resolved into components OP_2 and OR opposite in direction to these reactions. Let OL be normal to the back of the wall, and ON be normal to the plane BM. Then if motion is just about to occur, the angle NOR is equal to the angle of friction ϕ of earth upon earth, and LOP_2 is equal to the angle of friction ϕ' of earth upon masonry. Although ϕ'

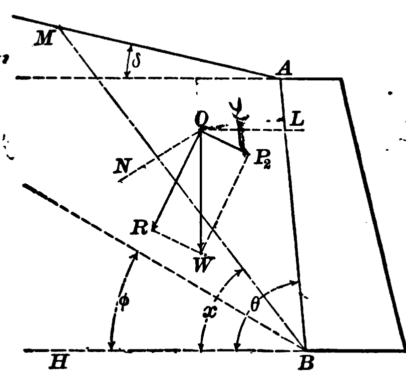


FIG. 9.

is perhaps in general greater than ϕ, it is customary to take it as the same, thus erring on the side of safety; accordingly $LOP_2 = \phi$.

Let W be the total weight of the earth in the prism BAM, and w its weight per cubic foot. Let P_2 represent the inclined resultant pressure against the wall. In the triangle ROW, the angle ROW is $x - \phi$, and ORW is $\theta + 2\phi - x$; hence

$$P_2 = W \frac{\sin (x - \phi)}{\sin (\theta + 2\phi - x)}.$$

The weight W for one unit in length of the wall is $w \times$ area $BAM \times 1$. The area of BAM equals $\frac{1}{2}BA \,.\, BM \,.\, \sin ABM$; the side BA is $h \div \sin\theta$, the angle ABM is $\theta - x$, and

$$BM = BA\,\frac{\sin BAM}{\sin AMB} = \frac{h \,.\, \sin(\theta - \delta)}{\sin\theta \,.\, \sin(x - \delta)}.$$

The value of W is thus expressed in terms of x and the given data, and P_2 becomes

$$P_2 = \tfrac{1}{2}wh^2\,\frac{\sin(\theta - \delta)\sin(\theta - x)\sin(x - \phi)}{\sin^2\theta\sin(x - \delta)\sin(\theta + 2\phi - x)}, \quad . \quad (27)$$

which gives the pressure due to any prism BAM.

The greatest possible value of P_2 is to be regarded as the actual value of the inclined pressure. By proceeding as in Article 8 it can be shown that this obtains when

$$\cot(\theta - x) = \cot(\theta - \phi) + \sqrt{[\cot(\theta - \phi) - \cot(\theta - \delta)]\,[\cot(\theta - \phi) + \cot 2\phi]}, \quad (28)$$

and that the maximum value is

$$P_2 = \frac{\tfrac{1}{2}wh^2\sin^2(\theta - \phi)}{\sin^2\theta\sin(\theta + \phi)\left(1 + \sqrt{\dfrac{\sin 2\phi \,.\, \sin(\phi - \delta)}{\sin(\theta + \phi)\sin(\theta - \delta)}}\right)^2}, \quad (29)$$

which is the general formula for the so-called inclined pressure and from which the results for all special cases can be deduced.

The greatest slope δ will be the natural slope ϕ. For this case the formula reduces to

$$P_2 = \frac{\frac{1}{2}wh^2 \sin^2(\theta - \phi)}{\sin^2\theta \sin(\theta + \phi)}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (30)$$

which is the pressure due to a bank of maximum slope against an inclined wall. If the back of the wall be vertical, $\theta = 90°$ and the expression takes the simple form $P_2 = \frac{1}{2}wh^2 \cos\phi$.

The most common case is that where the surface AM is horizontal; for this $\delta = 0$ and (29) becomes

$$P_2 = \frac{\frac{1}{2}wh^2 \sin^2(\theta - \phi)}{\sin^2\theta \sin(\theta + \phi)\left(1 + \sqrt{\frac{\sin 2\phi \sin\phi}{\sin(\theta + \phi)\sin\theta}}\right)^2}, \qquad (31)$$

which is the inclined pressure of a level bank of earth against an inclined wall. If in this $\theta = 90°$, there results the formula for a level bank of earth retained by a vertical wall.

$$P_2 = \frac{1}{2}wh^2 \frac{\cos\phi}{(1 + \sqrt{2}\sin\phi)^2}, \quad . \quad . \quad . \quad . \qquad (32)$$

which is the well-known expression deduced by PONCELET.

Problem 9. In formula (27) make $\theta = 90°$ and $\delta = 0°$. Then find the value of x which renders it a maximum, and deduce the corresponding value of P_2.

ARTICLE 10. GENERAL FORMULA FOR LATERAL PRESSURE.

Let a wall whose back is AB sustain a bank of earth BAM as in Figure 9. If the earth be loose, the weakest plane BM wil be that along which rupture is about to occur, so that the angle $NOR = \phi$, as in the two preceding articles. Let the resultant lateral pressure be designated by P, and let its direction make an angle z with the normal to the wall so that $LOP_2 = z$. By the same reasoning and methods as before used, it is found that the expression for the pressure due to any prism ABM and the value of cot $(\theta - x)$ which renders it a maximum are the same as given by (27) and (28) if the single term 2ϕ be replaced by $\phi + z$, and then results

$$P = \frac{\frac{1}{2}wh^2 \sin^2(\theta - \phi)}{\sin^2\theta \sin(\theta + z)\left(1 + \sqrt{\frac{\sin(\phi + z)\sin(\phi - \delta)}{\sin(\theta + z)\sin(\theta - \delta)}}\right)^2}, \quad (33)$$

which is a general formula for the lateral pressure in terms of the unknown angle z. If $z = 0$, the direction of P is normal to the back of the wall and (33) reduces to (23). If $z = \phi$, the pressure is inclined to the normal at the angle of friction and (33) reduces to (28).

From the above formula a number of theories of earth pressure can be deduced by making different assumptions with regard to the angle z. For instance, it seems to some authors a reasonable theory which makes the pressure upon a vertical wall parallel to the earth surface AM; for this case $\theta = 90°$, and $z = 90° + \delta - \theta$, and inserting these in (33) it reduces to

$$P_2 = \frac{\frac{1}{2}wh^2 \cos^2 \phi}{\cos \delta \left(1 + \sqrt{\frac{\sin(\phi + \delta)\sin(\phi - \delta)}{\cos \delta}}\right)^2}, \quad (34)$$

which is RANKINE'S formula for the lateral pressure against a vertical wall. In like manner several other formulas, more or less reasonable, can be established. But probably everything necessary for the practical engineer is given in Articles 8 and 9.

Problem 10. Deduce formulas for the earth pressure under the supposition that its direction is horizontal.

ARTICLE 11. COMPUTATION OF PRESSURES.

In computing the lateral pressure of earth from the above formulas it is customary to take h in feet and w in pounds per cubic foot; then the value of P will be in pounds per running foot of the wall. On account of the uncertainty in the data the trigonometric functions need be taken only to three or four decimal places, or, if logarithms be used, as will be found most convenient, a four-place table will be amply sufficient. The values of the pressures for several cases will now be compared, the walls all being 18 feet in vertical height, the earth weighing 100 pounds per cubic foot and having a natural slope $\phi = 34$ degrees. Here the value of $\frac{1}{2}wh^2$ is 16 200 pounds.

For a level bank of earth and a wall whose back slopes backward with the inclination $\theta = 80°$, formulas (25) and (31) give the pressures

$$P_1 = 3\,570, \qquad P_2 = 2\,590 \text{ pounds.}$$

For a level bank of earth and the back of the wall vertical, formulas (26) and (32) give

$$P_1 = 4\,580, \qquad P_2 = 4\,210 \text{ pounds.}$$

For a level bank of earth and the back of the wall sloping forward so that $\theta = 100°$, formulas (25) and (31) give

$$P_1 = 5\,760, \qquad P_2 = 5\,670 \text{ pounds.}$$

Here it must be remembered that the direction of P_1 is normal to the wall, while the direction of P_2 makes an angle of 34° with the normal to the wall.

For the same walls sustaining earth whose upper surface has the slope $\delta = 10$ degrees, the following values are found from formulas (23) and (29):

For $\theta = 80°$, $P_1 = 3\,920$, $P_2 = 3\,400$ pounds.
For $\theta = 90°$, $P_1 = 5\,080$, $P_2 = 4\,960$ pounds.
For $\theta = 100°$, $P_1 = 6\,469$, $P_2 = 6\,480$ pounds.

For the same walls sustaining earth whose upper surface has the angle of repose $\delta = 34°$, formulas (24) and (30) give:

For $\theta = 80°$, $P_1 = 8\,780$, $P_2 = 9\,460$ pounds.
For $\theta = 90°$, $P_1 = 11\,130$, $P_2 = 13\,430$ pounds.
For $\theta = 100°$, $P_1 = 14\,160$, $P_2 = 19\,380$ pounds.

A comparison of the above values shows that the pressure increases both with θ and δ. For a level bank of earth the values of P_1 are less than those of P_2, but for a large value of δ the values of P_2 become greater than those of P_1. Whether the true thrust against the wall is P_1 or P_2, or some intermediate value, cannot be determined theoretically, and hence the best procedure for the engineer will be to use those values which are the most unfavorable to stability.

For the case of water $\phi = 0$ and $\delta = 0$, and all the formulas for pressures reduce to

$$P = \tfrac{1}{2}wh^2 \div \sin\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

in which w is the weight of a cubic foot of water, or $62\frac{1}{2}$ pounds. The pressure of water against a wall 18 feet in vertical height will hence be 10 280 pounds when $\theta = 80$ degrees, 10 125 pounds when $\theta = 90$ degrees, and 10 280 pounds when $\theta = 100$ degrees, its direction being always normal to the back of the wall.

Problem 11. Compute the pressures against a wall 9 feet in vertical height for earth weighing 100 pounds per cubic foot and having an angle of repose $\phi = 34$ degrees, (*a*) for the case when $\delta = 10$ degrees and $\theta = 80$ degrees; (*b*) for the case when $\delta = 20$ degrees and $\theta = 80$ degrees.

ARTICLE 12. THE CENTRE OF PRESSURE.

For all the above cases the formulas for the resultant lateral pressure of the earth may be written

$$P = \tfrac{1}{2}wh^2 . k,$$

in which k is a function of the angles θ, δ, and ϕ. If y represent any vertical depth measured downward from the top, the resultant pressure on the part of the wall corresponding to this depth is

$$P = \tfrac{1}{2}wy^2 . k,$$

which shows that the resultant pressure varies as the square of the height of the wall. The pressure per square unit at any point on the wall varies, however, directly as the height, for if y be increased by dy the increase in P is dP or $wkydy$, and the pressure per square unit over the area $1 \times dy$ is

$$\frac{dP}{dy} = wky.$$

The laws governing the distribution of earth pressures are hence the same as for water, the unit-pressure at any point varying as the depth, and the total pressure as the square of the depth.

The point where the resultant pressure P is applied to the wall is called the centre of pressure, and this is at a vertical distance from the top of the wall equal to two-thirds its height. This may be proved by Figure 10, which gives a graphical representation of the pressure against the back of the wall, the unit-pressures falling into a triangular shape, since each is proportional to its distance below the top. The point of

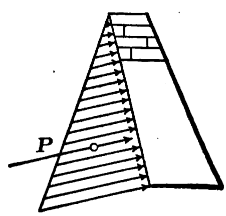


FIG. 10.

application of the resultant pressure P hence passes through the centre of gravity of this triangle and cuts the back at two-thirds its length from the top.

For another proof the principle of moments may be used. Let y be any vertical distance from the top, and y' the vertical distance from the top to the centre of pressure. Then taking the top of the back of the wall as the origin of moments,

$$P.y' = \Sigma dP.y.$$

Inserting in this the values of P and dP given above and integrating betwen the limits $y = 0$ and $y = h$, there results

$$y' = \tfrac{2}{3}h; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (36)$$

that is, the centre of pressure is at a vertical distance $\frac{2}{3}h$ below the top of the wall, or at $\frac{1}{3}h$ above the base.

Problem 12. Let a level bank of earth have a load of q pounds per square foot upon the surface AM. Show that the resultant normal pressure due to both bank and load is

$$P_1 = (\tfrac{1}{2}wh^2 + qh)\frac{\sin^2 \frac{1}{2}(\theta - \phi)}{\sin\theta \,.\, \sin^2 \frac{1}{2}(\theta + \phi)}, \quad . \quad . \quad . \quad (37)$$

and that the depth of the centre of pressure below the top of the wall is

$$y' = \frac{2wh + 3q}{3wh + 6q} \cdot h. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Find the position of the centre of pressure when $h = 18$ feet, $w = 100$ pounds per cubic foot, and $q = 300$ pounds per square foot.

CHAPTER III.

INVESTIGATION OF RETAINING WALLS.

ARTICLE 13. WEIGHT AND FRICTION OF STONE.

The lateral pressure of the earth against a retaining wall tends to cause failure in two ways, namely, by sliding and by rotation. This tendency is resisted by the friction between the stones and by the weight of the wall. The following table gives average values of the unit-weights and the coefficients of friction for different kinds of masonry:

Kind of Masonry.	Coefficient of Friction.	Angle of Friction.	Weight.	
			Pounds per cubic foot.	Kilos per cubic meter.
Limestone and Granite:				
Ashlar Masonry........	0.6	31°	165	2640
Large Mortar Rubble....			150	2400
Small Dry Rubble......			125	2000
Sandstone:				
Ashlar Masonry........	0.6	31°	150	2400
Large Mortar Rubble....			130	2100
Small Dry Rubble.......			110	1760
Coarse Brickwork.........	0.65	33°	100	1600

In the investigation of masonry walls the influence of the mortar is generally neglected, on account of its uncertain character and because the error is then on the side of safety. The above coefficients of friction are hence stated for dry masonry, and will probably be somewhat increased when mortar is used. For rubble masonry the coefficient of friction is often somewhat higher than for ashlar; but its value is so uncertain that no figure is given in the table.

The coefficient of friction of stone upon stone is determined by placing two plane surfaces together and then gradually inclining the surface of contact until the upper stone begins to slide upon the lower. The angle made by the plane with the horizontal is the angle of friction, and its tangent is the coefficient of friction. as shown by equation (2).

The word "batter" means the inclination of the face or back of a wall, measured by the ratio of its horizontal to its vertical projection, or in inches of horizontal projection per foot of vertical height. Let θ be the angle of inclination of the back of the wall to the horizontal, as in Figure 8. Then $\cot\theta$ is the batter of the back, and the values of θ, $\sin\theta$, and $\cos\theta$ for different batters are given in the following table. If the back of the wall leans backward, θ is less than 90 degrees, and $\cos\theta$ and $\cot\theta$ are positive; if it leans forward, θ is greater than 90 degrees and $\cos\theta$ and $\cot\theta$ are negative; $\sin\theta$ is positive in both cases. These values will be useful in many computations.

Batter in inches per foot.	Angle θ for Backward Batter.	Angle θ for Forward Batter.	sin θ.	cos θ.	Batter cot θ.
0	90° 00′	90° 00′	1.0000	0.0000	0.0000
½	87 37	92 23	0.9991	0.0461	0.0417
1	85 14	94 46	0.9965	0.0831	0.0833
1½	82 52	97 08	0.9923	0.1239	0.1250
2	80 32	99 28	0.9864	0.1645	0.1667
2½	78 14	101 46	0.9790	0.2039	0.2083
3	75 58	104 02	0.9702	0.2425	0.2500
3½	73 45	106 15	0.9600	0.2797	0.2916
4	71 34	108 26	0.9487	0.3162	0.3333
5	67 29	112 31	0.9239	0.3828	0.4144
6	63 26	116 34	0.8944	0.4472	0.5000

Problem 13. Compute the values of sin θ, cos θ, tan θ, and cot θ for a batter of 4½ inches per foot; also for a batter of 5½ inches per foot.

ARTICLE 14. GENERAL CONDITIONS REGARDING SLIDING.

A retaining wall may fail by sliding on its base or on some joint above the base. When a wall is just on the point of failure it is in the state of equilibrium, that is, the weight of the wall just balances the pressure and reaction of the earth. The proper state of a wall is, of course, stability; and failure brings disgrace upon its designer.

The degree of stability of a wall against sliding may be indicated by a number called the factor of security which ranges in value from unity to infinity. This factor will be designated by n; when $n = 1$ equilibrium exists and the wall will fail;

when $n > 1$ the wall is stable and its degree of stability varies with n; when $n = \infty$ the highest possible state of stability exists.

The analytical conditions of equilibrium and stability for the case of sliding are the following. Let Figure 11 represent two bodies having a plane surface of contact, N the total force

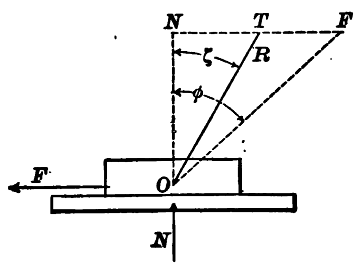


FIG. 11.

normal to that plane, F the total force parallel to it, and f the coefficient of friction between the surfaces. Then the condition of equilibrium is, as in (1),

$$F = fN, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

and the condition of stability is

$$F < fN \qquad \text{or} \qquad nF = fN, \quad . \quad . \quad . \quad . \quad (40)$$

in which n is a number greater than unity called the factor of security. The equation (40) may be used for the discussion of all cases of sliding, F and N being the sum of the components in the directions parallel and normal to the plane of all the forces exerted by one body on another. If R be the resultant of all these forces and ζ be the angle which it makes with the normal ON, the value of F is $R \sin \zeta$, and that of N is $R \cos \zeta$. Inserting this in (40) it becomes

$$n \tan \zeta = f, \qquad (41)$$

which is another form of the condition of stability against sliding.

The graphical conditions of equilibrium and stability for sliding are simple. In Figure 11 let ON be normal to the plane of contact and NOF be the angle of friction ϕ, that is, the angle whose tangent is f. Let R make an angle ζ with the normal ON, then equilibrium obtains when ζ equals ϕ, and stability occurs if ζ is less than ϕ. Draw NF parallel to the plane of contact, and let T be the point where it intersects the line of direction of R. The position of T indicates the degree of stability against sliding; if the distances NT and NF be determined, the factor of security is the ratio of the latter to the former, or

$$n = \frac{NF}{NT}; \qquad (42)$$

for, it is seen that this formula is the same as (41), NF being f, and NT being $\tan \zeta$ if the distance ON be unity, and their ratio being the same as these tangents whatever be the length of ON.

Problem 14. A plane surface is inclined at an angle of 40° to the horizontal, and on it is a block weighing 125 pounds, against which, to prevent it from sliding, a horizontal force of 300 pounds acts. If the angle of friction of the block upon the plane is 18°, compute the factor of security against sliding.

ARTICLE 15. GRAPHICAL DISCUSSION OF SLIDING.

Let Figure 12 represent the section of a wall whose dimensions and weight are given. Let *BC* be any joint extending through the wall, and let *P* be the lateral pressure of the earth above *B*. It is required to investigate the security of the wall against sliding.

The pressure *P* is applied on the back of the wall at one third of its height above *B*, and its direction depends on the hypothesis adopted in its computation; if Article 8 is used, it is normal to the back of the wall; if Article 9, it is inclined at an angle equal to the angle of natural slope of the earth.

A drawing of the given cross-section is made to scale, and its centre of gravity found: this is *G* in the figure. The area of this cross-section is then determined and called *A*; if this be multiplied by *v*, the weight of a cubic unit of masonry, the product is *V*, the weight of a wall one unit in length, or $V = vA$.

Through *G* a vertical line is drawn, and the direction of *P* is produced to intersect this in *O*. Lay off *OP* to scale equal to the earth pressure *P*, and *OV* equal to the weight of the wall, *V*. Complete the parallelogram of forces *OPRV*, thus finding *OR* as the resultant of *P* and *V*.

Produce *OR* to meet the joint *BC* in *T*. Through *O* draw *ON* normal to *BC*, and then draw *OF*, making the angle *NOF* equal to the angle of friction of stone upon stone. This completes the graphical work.

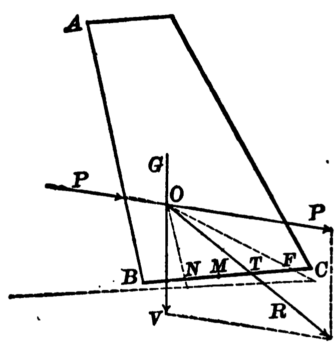


FIG. 12.

If the point *T* falls between *N* and *F*, the wall will not fail by sliding, and its stability is the greater the nearer *T* is to *N*. If *T* coincides with *F*, the wall is just on the point of sliding along the joint *BC*, and much more so is this the case if *T* falls beyond *F*. As explained in the last Article, a numerical expression of the degree of stability can be obtained by dividing the distance *NF* by *NT*, or if *n* be the factor of security against sliding,

$$n = \frac{NF}{NT}.$$

This becomes unity when *NT* equals *NF*, and infinity when *NT* is zero, the first value indicating the failure of the wall and the second the greatest possible degree of stability against sliding. It is recommended that for first-class work *n* should not be less than 3.0, and fortunately it is always easy in building a wall to make its value greater than this by properly inclining the joints (Article 23).

The above method applies either to the base of the wall or to any joint that extends through it, whether the joint be horizontal or inclined. Owing to the uncertainty regarding the weight and angle of repose of the earth, the direction of *P*, and the angle of friction of the stone, it will not always be possible to obtain values of the factor of security which are perfectly satisfactory. Still the investigation will generally determine if danger exists, and of course unfavorable values of the data should be used in the analysis. If the wall have no joints extending through it, an analysis for sliding need not be made.

Problem 14. Prove that the centre of gravity of a quadrilateral *abcd* can be found as follows: Draw the diagonal *ac* and bisect it in *e*; join *be*, and take *ef* equal to $\frac{1}{3}$*be*; through *f* draw *fk* parallel to *bd*. Draw the other diagonal *bd* and bisect it in *g*; join *ge*, and take *gh* equal to $\frac{1}{3}$*eg*; through *g* draw *gk* parallel to *ac*. The centre of gravity is at *k*, the intersection of *fk* and *gk*.

ARTICLE 16. ANALYTICAL DISCUSSION OF SLIDING.

Let $ABCD$ represent the cross-section of a wall whose dimensions and weight are given, θ being the inclination of its back to the horizontal. Let BC be any joint extending through the wall, and α its inclination to the horizontal. Let P be the lateral pressure of the earth above this joint, and z the angle between its direction and the normal to the wall;

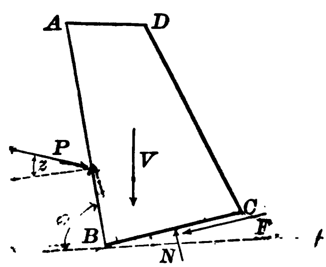


FIG. 13.

if P be the pressure computed by the formulas in Article 8, the value of z is simply zero; if by those in Article 9, its value is the angle of repose of the earth; if z be assumed at any intermediate value, P is computed from (33). Let V be the weight of the wall. It is required to investigate the degree of stability against sliding.

Let F and N be the sum of the components of P and V respectively parallel and normal to the joint, and f the coefficient of friction. Then if n be the factor of security, $nF = fN$, and

$$n = \frac{fN}{F}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Now by resolving P and V parallel and normal to the joint there is found

$$\left.\begin{aligned} F &= P \sin(\theta + \alpha + z) - V \sin\alpha, \\ N &= V \cos\alpha - P\cos(\theta + \alpha + z); \end{aligned}\right\} \quad . \quad . \quad . \quad (44)$$

and if these be inserted in (43), the value of n is expressed in terms of the given data. The entire analytical discussion of the sliding of a wall along a joint consists in computing n from these formulas. If n is greater than 3, the security against sliding is ample; if n is less than 3, the wall does not have proper security for first-class work; if $n = 1$, failure will occur.

For example, consider a sandstone wall 18 feet high, 3 feet wide at the top and 6 feet wide at the base, the back being vertical. The weight of the masonry is taken at 140 pounds per cubic foot, and the coefficient of friction on the horizontal joint at the base is 0.5. This wall supports a level bank of earth weighing 100 pounds per cubic foot and having an angle of natural slope of 34 degrees. It is required to find its factor of security against sliding.

First, let the pressure P and its direction be taken from Article 8. Here $h = 18$ feet, $w = 100$, $\theta = 90°$, $\phi = 34°$,

$\delta = 0°$, and $z = 0°$. Then by formula (26) there is computed $P = 4580$ pounds. The weight of the wall is

$$V = 140 \times 18 \times 4\tfrac{1}{2} = 11340 \text{ pounds.}$$

Now since $\alpha = 0°$, $F = 4580$ and $N = 11340$, hence the factor of security is

$$n = \frac{11340 \times 0.5}{4580} = 1.2,$$

which indicates a very low degree of stability.

Secondly, let the pressure P and its direction be taken from Article 9. Here $z = 34°$, and using formula (32) there is found $P = 4210$ pounds. V is 11340 pounds as before. From (44),

$$F = 4210 \sin (90° + 34°) = 3490,$$
$$N = 11340 - 4210 \cos (90° + 34°) = 13690,$$

and then from (43) there results the factor

$$n = \frac{0.5 \times 13690}{3490} = 1.9,$$

which indicates a degree of stability too low for first-class work.

Unfortunate indeed it is that the theory of earth pressure is not sufficiently explicit to determine the exact value and direction of P. He who believes the theory of Article 8 must conclude that this wall is in a very dangerous condition and almost about to slide; he who defends the theory of Article 9 might conclude that it is not in great danger, and that its degree of security is fair. It is well, however, not to forget that the given data are liable to variations fully as serious as the defects in the theory. Imagine a heavy rainfall to increase w and decrease ϕ; this causes P to become larger, and as F usually would be smaller in wet weather, it is seen that the degree of stability of the wall would be greatly diminished. If the factor of security be computed for both theories as is done above, and the variation in the data be regarded, a fair conclusion can generally be made regarding the security of the wall. The effect of the variable data, however, is often so great that a ripe judgment, based upon experience, may be more reliable than computations.

Problem 16. Owing to a heavy rainfall the earth behind the above wall is increased in weight to 120 pounds per cubic foot and the angle of natural slope is decreased to 32 degrees, while the coefficient of friction at the base of the wall becomes 0.45. Compute the factor of security of the wall against sliding, (*a*) using the theory of Article 8, and (*b*) using that of Article 9.

ARTICLE 17. GENERAL CONDITIONS REGARDING ROTATION.

Let Figure 14 represent two bodies having the plane of contact BC. Let M be the middle point of BC. Let R be the resultant of all the forces which each body exerts on the other, and let T be its point of application on BC. It is clear that rotation or overturning will instantly occur if T falls without BC, that equilibrium obtains if T coincides with C,

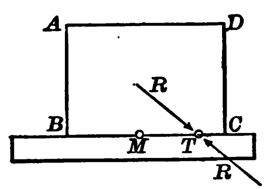


FIG. 14.

and that stability, more or less secure, will result if T falls within BC. The nearer the point of application T is to the middle of the base M the greater is the degree of stability against rotation.

To investigate the degree of security of a given wall against rotation it is only necessary to find the distance MT either graphically or analytically. Let n be the factor of security of the wall, then

$$n = \frac{MC}{MT}. \quad \ldots\ldots\ldots \quad (45)$$

If MT equals MC, the value of n is unity and failure by rotation is about to occur; if MT is less than MC, the value of n is greater than unity and the wall is more or less stable; if MT is zero, n is infinity and the wall has the greatest possible degree of stability.

The factor of security n should not have a value less than three for proper stability. To demonstrate this, consider the distribution of pressures in a joint as represented in Figure 15.

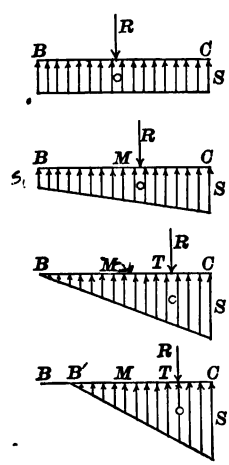


FIG. 15.

In the first diagram the resultant pressure R is applied at the middle of BC; here the pressure will be uniformly distributed over the joint, and the unit-stress S_1 at B will be equal to the unit-stress S at C. In the second diagram the resultant R is applied so that MT has a small value; then the pressure is not uniformly distributed over the joint, but the unit-stress S_1 at B becomes smaller than in the first diagram, while the unit-stress S at C becomes greater, and the unit-stresses between B and C are taken as varying proportionally. In the third diagram the distance MT is such that the unit-stress at B becomes zero; this occurs when CT is one-third of CB (since the line of direction of R passes through the centre of gravity of the stress triangle) or when MT is one-third of MC. In the last diagram MT has become greater than one-third of MC, so that the pressure is only distributed over CB' and the portion BB' is either brought into tension or the joint opens. As masonry joints cannot take tension this last is a dangerous condition. Therefore the ratio of MC to MT, or the factor of security, should not be less than 3.0.

If the joint BC be divided into three equal parts, so that $BD = DE = EC$, the portion DE is called the "middle third," and the above requirement is otherwise expressed by saying that for proper security against rotation the resultant of all the forces above any joint must be within the middle third of that joint.

Problem 17. In Figure 15 let BC be horizontal, and let $ABCD$ be a cubical block weighing 625 pounds. Compute the factor of security against rotation when a horizontal force of 250 pounds is applied at A.

ARTICLE 18. GRAPHICAL DISCUSSION OF ROTATION.

Let Figure 12 represent the cross-section of a wall whose dimensions and weight are given. Let BC be any joint extending through the wall, and let P be the lateral pressure of the earth above B. It is required to investigate the security of the wall against rotation.

The pressure P is applied on the back of the wall at one-third of its height above B (Article 12), and its direction is either normal to the wall (Article 8), inclined to the normal at the angle of natural slope of the earth (Article 9), or it has a direction between these two limits (Article 10).

A drawing of the given cross-section is made to scale, and its centre of gravity found; this is at G. The area of this cross-section is next determined and called A; then the weight of the wall for one unit in length is $V = vA$, where v is the weight of the masonry per cubic unit.

Through G draw a vertical line and produce P to intersect it in O. Lay off OP to scale equal to the earth pressure P, and OV equal to the weight V. Complete the parallelogram of forces $OPRV$, thus finding OR as the resultant of P and V. Produce OR to meet the joint BC in T. Mark M as the middle point of BC, and measure MT and MC. This completes the graphical work.

If T falls at C, the wall is on the point of rotation; and if at M, it has the highest possible degree of stability. If BC be divided into three equal parts and T is found within the middle one, the wall has proper security against rotation. If it falls without the middle third, it is deficient in security (Article 17). Dividing MC by MT the factor of security is found, or

$$n = \frac{MC}{MT}.$$ [45] [3] Rotation

If this is unity or less, the wall fails; if it be smaller than 3, the wall is stable but not secure; if it be greater than 3, the degree of security is sufficient as far as rotation alone is concerned; if it be infinity, nothing more can be desired.

By this method but one construction is needed for the investigation of a wall against both sliding and rotation. It will usually be found for ordinary cases that the factor of security against rotation is least for the base of the wall or for the lowest joint. For the general discussion the base of the wall is drawn inclined in Figure 12, but in the actual drawing it will be best to take it as horizontal.

Problem 18. Let a wall with vertical back support a level bank of sand weighing 100 pounds per cubic foot and having an angle of repose of 34 degrees. Let the top of the wall be 2 feet thick, its base 7.57 feet, its vertical height 20 feet, and its weight per cubic foot 165 pounds. Determine the factors of stability against sliding and rotation for the horizontal base, taking the earth pressure from Article 8.

ARTICLE 19. ANALYTICAL DISCUSSION OF ROTATION.

Let *ABCD* be a cross-section of a wall with vertical back, *AB* being 24 feet, the top *AD* being 3 feet and the base *BC* being 8 feet. Let the weight per cubic foot of the masonry

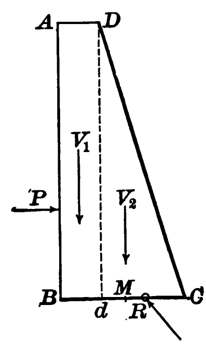


FIG. 16.

be 150 pounds, and let it be required to determine the factor of security against rotation for a horizontal earth pressure *P* of 4000 pounds.

Let *T* be the point where the resultant pierces the base, and let *CT* be represented by *t*; then the factor of security is

$$n = \frac{MC}{MT} = \frac{4}{4 - t},$$

in which t is to be determined. To do this, drop Dd perpendicular to BC, dividing the cross-section into a rectangle of weight V_1 and a triangle of weight V_2. The value of V_1 is $150 \times 3 \times 24$ or 10,800 pounds, and its horizontal distance from T is $(6\frac{1}{2} - t)$ feet. The value of V_2 is $150 \times 5 \times 12$ or 9000 pounds, and its horizontal distance from T is $(\frac{2}{3} \times 5 - t)$ feet. The lever arm of P with reference to T is 8 feet, and as R passes through T its lever arm is 0. Then the equation of moments is

$$8000 \times 8 = 10800(6\tfrac{1}{2} - t) + 9000(3\tfrac{1}{3} - t),$$

from which t is found to be 1.83 feet, and then the factor of security against rotation is

$$n = \frac{4}{2.17} = 1.9,$$

which is not sufficient for proper stability.

A general discussion applicable to any trapezoidal cross-section will now be given. Let h be the vertical height of the wall, a the thickness of the top AD, b the thickness of its base BC, and v its weight per cubic foot. Let θ be the angle which the back of the wall makes with the horizontal, and z the angle which the earth pressure P makes with the normal to the back of the wall. The point of application of P is at a vertical height of $\frac{1}{3}h$ above B.

Let V be the weight of the wall acting through the centre of gravity of the cross-section, and let S be the point where its direction cuts the base. Let R be the resultant of P and V

acting at some point T on the base. Let s represent the distance BS_1, and t the distance CT. Let the point T be taken as a centre of moments, and let the lever-arm of P with

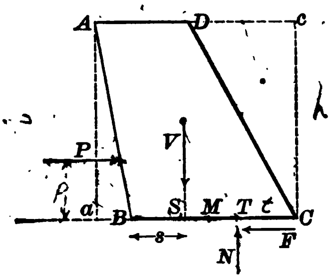


FIG. 17.

reference to it be p. The lever-arm of V is $b - s - t$, and that of R is zero. Then the equation of moments is

$$Pp = V(b - s - t), \quad \quad (46)$$

which is the fundamental formula for the investigation of retaining walls. This may be written

$$Pp = Vb - Vs - Vt, \quad \quad (47)$$

which is sometimes a more convenient form for use, since Vb and Vs can be treated like single quantities.

To investigate a wall, the factor of security n is to be determined. From formula (45),

$$n = \frac{MC}{MT} = \frac{\frac{1}{2}b}{\frac{1}{2}b - t}, \quad \quad (48)$$

and n will be known as soon as t is found. To do this the value of p is expressed in terms of t, thus:

$$p = \tfrac{1}{3}h\frac{\cos z}{\sin\theta} + (b - t)\cos(\theta + z), \quad . \quad . \quad . \quad . \quad (49)$$

and this being inserted in (47) there is deduced

$$t = \frac{Vb - Vs - \tfrac{1}{3}Ph\dfrac{\cos z}{\sin\theta} - Pb\cos(\theta + z)}{V - P\cos(\theta + z)}. \quad . \quad (50)$$

In this formula V is the weight of one unit in length of the wall, or, for a trapezoid,

$$V = \tfrac{1}{2}vh(a + b), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

and Vs is the moment of that weight with respect to the inner edge B of the base. By considering the trapezoid $ABCD$ as the difference between the rectangle $AaCc$ and the two triangles AaB and CcD, this is found, by the help of the principles of statics, to be

$$Vs = \tfrac{1}{6}vh\left(a^2 + ab + b^2 - (2a + b)h\cot\theta\right), \quad . \quad (52)$$

and, dividing by V, the distance s can be determined if it should be required. To investigate a wall, formulas (52) and (51) are first used, then (50), and lastly (45).

As an example, let the following data be taken: A sandstone wall retaining a level bank of earth; $\phi = 34$ degrees, $w = 100$ pounds per cubic foot, $h = 18$ feet, $a = 2$ feet, $b = 5$ feet, $\theta = 80$ degrees, $v = 140$ pounds per cubic foot. The value of P, from Article 8, is 3570 pounds, z being zero. The weight V is found by (51) to be 8820 pounds. Vs is found by (52) to be 4405 pounds-feet. These inserted in (50) give $t = 1.75$ feet, which, being greater than one-third the base, shows proper stability; and lastly, from (48), the factor of security is $n = 3.3$.

It will be interesting to test the same wall by the pressure theory of Article 9, where, z being 34 degrees, P is 2590 pounds. All other data being the same as before, there is found from (50) the value $t = 3.23$ feet, which is more than one-half the base, so that T in Figure 17 lies between M and B, and the tendency to rotation about B is greater than that about C.

Problem 19. Compute the factor of security against rotation for the data given in Problem 18.

ARTICLE 20. COMPRESSIVE STRESSES ON THE MASONRY.

As a general rule, the working compressive stress upon the base of masonry walls should not exceed 150 pounds per square inch in first-class work. A tower 150 feet in height will produce this pressure on its base if the masonry weighs 144 pounds per cubic foot.

The total normal pressure N upon the horizontal base of a retaining wall will be given by (44), making $\alpha = 0°$, or

$$N = V - P \cos (\theta + z), \quad . \quad . \quad . \quad . \quad . \quad (53)$$

in which P is the earth-pressure acting at an angle z with the normal to the back of the wall, V the weight of the wall, and θ the angle which its back makes with the horizontal. If P is computed by Article 8, the angle z is zero; if by Article 9, its value is ϕ, the angle of repose of the earth. For any ordinary case $\cos (\theta + z)$ is a small fraction, and in most cases it is a sufficient practical approximation to regard N as equal to V.

The compressive stresses upon the base BC (Figure 17) will be regarded as caused by the vertical pressure N alone. N is evidently the vertical component of the resultant R. The horizontal component of R produces shearing stresses along the base which are supposed not to increase the compressive stresses. The distribution of the compression over the base will then be similar to that shown in the diagrams of Figure 15, and will depend upon the position of the point in which R cuts the base.

If the resultant cuts the base at its middle point (as in the first diagram of Figure 15), the compression due to N is uniform over the area $b \times 1$ square feet, and

$$S = \frac{N}{b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

is the compressive stress in pounds per square foot.

If the resultant is applied at the limit of the middle third (as in the third diagram of Figure 15), the unit-stress at the edge B is zero, that at the middle is the average value given by (54), and the greatest stress at the toe C is double this average value, or

$$S = 2\frac{N}{b}. \quad \quad (55)$$

If the resultant is applied without the middle third at a distance t from the edge C (as in the last diagram of Figure 15), the compression is distributed only over the distance $3t$, so that

$$S = 2\frac{N}{3t}. \quad \quad (56)$$

gives the stress in pounds per square foot.

The case where R cuts the base within the middle third at a distance t from C (as in the second diagram of Figure 15) remains to be considered. Let S be the greatest unit-stress at C, and S_1 be the least unit-stress at B. Then the unit-stress at the middle of the base is equal to the average unit-stress, or

$$\frac{S + S_1}{2} = \frac{N}{b};$$

and as N is applied opposite the centre of gravity of the stress-trapezoid, the value of t is

$$t = \frac{S + 2S_1}{S + S_1} \cdot \frac{b}{3}.$$

Now eliminating S_1 from these two equations, there results

$$S = \frac{2N}{b}\left(2 - 3\frac{t}{b}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

which is the greatest unit-stress, namely, that at the toe C. As N is in pounds for one foot in length of wall, and b is in feet, these formulas give compressive stresses in pounds per square foot, and dividing by 144 the values in pounds per square inch are found.

For example, take the wall of the last article, where $h = 18$ feet, $a = 2$ feet, $b = 5$ feet, $\theta = 80$ degrees, $z = 0°$, $P = 3570$ pounds, $V = 8820$ pounds, and $t = 1.75$ feet. In (44) the value of α is $0°$, and N is found to be 8665 pounds. Then from (57) the greatest compression is 22.9 pounds per square inch, which is a low value even for inferior work.

For ordinary walls a sufficiently exact computation of the unit-stress S may be made by taking V for the value of N. Thus for the above case $V = 8820$, and from (57) $S = 23.3$ pounds per square inch. When $z = 0°$ and $\alpha = 0°$, formula (44) gives $N = V - P \cos \theta$, which differs but little from $N = V$ when θ is near $90°$.

If there be no pressure behind a wall, the point T coincides with S (Figure 17). Then the normal pressure V produces the greatest unit-stress at B, whose value is given by one of the formulas,

$$S = 2\frac{V}{3s}, \qquad \text{or} \qquad S = \frac{2V}{b}\left(2 - 3\frac{s}{b}\right), \quad . \quad . \quad . \quad (58)$$

according as the distance s is less or greater than one-third of b.

According to the theory here presented, the vertical component of R alone produces compression on the base of a retaining wall, while the horizontal component is exerted in producing a shearing stress. This theory has defects; but upon it has been based the design of structures more important than retaining walls.

Problem 20. Let the back of the wall be inclined forward at a batter of 2 inches per foot, and let the normal pressure of the earth be $P = 22,760$ pounds. Let its height be 36 feet, the top thickness 6 feet, the base thickness 18 feet, and the weight per cubic foot 150 pounds. Compute the greatest compressive stress on the base.

CHAPTER IV.

DESIGN OF RETAINING WALLS.

ARTICLE 21. DATA AND GENERAL CONSIDERATIONS.

When a retaining wall is to be designed for a particular location the character of the earth to be supported is known and also the height of the wall. The data then are: w the weight per cubic foot of the earth, ϕ its angle of repose, δ the angle of inclination of its surface, and h the height of the proposed wall.

The thickness of the top of the wall, a, is first assumed. In doing this practical considerations will generally govern rather than theoretical ones. Theory indicates, as will be seen in Article 24, that the thinner the top of the wall the less is the quantity of material required; but theory supposes the earth to be homogeneous and takes no cognizance of the action of frost. Experience, however, teaches that the freezing earth near the top of the wall exerts a marked lateral pressure which can only be counteracted by a substantial thickness. To possess proper stability against the action of frost and the weather a wall should not have a top thickness less than two feet. Usually when the height of a wall varies, as in a railroad cut, the top has the same thickness throughout. If a wall be only a foot high, its thickness should not be less than two feet, else in a few years the frost will push it over. Even in latitudes where frost is rare this rule is a good one to follow.

The engineer will next decide upon the batter of the back of the wall, or upon the value of θ, the angle between the back and the horizontal. In doing this he must have regard to the batter which the front of the wall will have, and to the theory of economy of material set forth in the following articles. In construction the back of the wall will be left rough or built in a series of steps, so that θ need be taken only to the nearest degree of the average inclination.

The pressure of the earth can now be computed by the proper formula of Chapter II. The theory of Article 8 which supposes its direction to be normal to the back of the wall is, in general, to be preferred, because in practice the earth is tamped against the wall so that there can be little tendency to slide along it. Article 8 demands a heavier wall than Article 9, and is thus on the side of safety. In our opinion Article 8 gives the pressure of earth against a wall which stands firmly with a high degree of stability, and Article 9 gives the pressure of the earth when motion or failure is about to begin. As walls are designed to stand and not to fail, the engineer should be careful in erring on the side of safety.

Therefore in this chapter the lateral pressure P will usually be taken normal to the back of the wall, so that in all previous formulas the angle z is zero.

The next procedure is to determine the thickness of the base so that the wall may have proper security against rotation; how this is done the following Article will show. If the front of the wall thus designed does not have the desired batter, a change can be made in the value of θ and the work be repeated. Then it will be well to test the work by determining graphically (Article 18) the factor of security against rotation. Lastly, the question of sliding must be considered and proper security against it be provided (Article 23). The practical points regarding the coping, the frost batter near the top of the back of the wall, the weep holes, the foundation, the drainage ditches, the quality of the masonry, and the details of construction will, of course, receive full attention and be fully set forth in the drawings and specifications.

Problem 21. If β be the angle which the front of the wall makes with the horizontal, prove that $b - a$ equals h ($\cot \beta - \cot \theta$). Find the batter of both back and front in inches per foot when $b = 5$ feet, $a = 2$ feet, $h = 18$ feet, and $\theta = 80$ degrees.

ARTICLE 22. COMPUTATION OF THICKNESS.

The discussion in Article 19 furnishes the following fundamental equation for the stability of any wall against rotation:

$$Pp = Vb - Vs - Vt.$$

To apply this to the determination of the thickness of the base of a trapezoidal wall the values of p, V and Vs are inserted from (49), (51) and (52) and t is made $\frac{1}{3}b$, thus giving a factor of security of 3.0 against rotation (Article 17). The value of the angle z is taken as zero because the earth pressure is computed from Article 8 under that supposition. Then results

$$P_1\left(2b\cos\theta + \frac{\frac{1}{3}h}{\sin\theta}\right) = \tfrac{1}{2}vh(b^2 + ab - a^2 + bh\cot\theta + 2ah\cot\theta), \quad (59)$$

and the solution of this equation with respect to b gives

$$b = -A + \sqrt{B + A^2}, \quad . \quad . \quad . \quad . \quad . \quad (60)$$

in which A and B have the values

$$A = \tfrac{1}{2}(a + h\cot\theta) - \frac{2P_1\cos\theta}{vh},$$

$$B = \frac{2P_1}{v\sin\theta} + a^2 - 2ah\cot\theta,$$

from which the base thickness can be computed for any given data.

When the value of θ is 90 degrees this takes the simple form

$$b = -\tfrac{1}{2}a + \sqrt{\frac{2P}{v} + \frac{5a^2}{4}}, \quad . \quad . \quad . \quad . \quad (61)$$

which is the formula for the base thickness of a wall with vertical back.

In these formulas P_1 is the earth pressure computed by Article 8, h the vertical height of the wall, a the thickness of its top, θ the angle at which the back is inclined to the horizontal, v the weight of a cubic foot of masonry, and b is the thickness of the base which gives the wall a factor of security of 3.0 against rotation, the resultant R then cutting the base at the limit of the middle third. For all joints above the base the factor of security will then be greater than 3.0.

For example, let a wall with vertical back be 20 feet high, sustaining a level bank of sand which weighs 100 pounds per cubic foot and has a natural slope of 34 degrees. Let the masonry be 165 pounds per cubic foot and the top of the wall be 2 feet in thickness. It is required to find the thickness of the base BC (Figure 16). From formula (26) the pressure of the earth is 5650 pounds. Then from (61)

$$b = -1 + \sqrt{\frac{2 \times 5650}{165} + 5} = 7.57 \text{ feet},$$

which gives a cross-section whose area is $\frac{1}{2}(2+7.57)16$, or 95.7 square feet.

As a second example take the same wall except that the back is inclined backward so that θ is 80 degrees. Here the value of P_1 is found from (25) to be 4410 pounds. Then $a = 2$, $h = 20$, $\cos\theta = +0.1736$, $\cot\theta = +0.1763$, $v = 165$, whence from (60) $A = 229$, $B = 44.2$, and $b = 4.45$ feet, which gives a cross-section whose area is 64.5 square feet. The advantage of inclining the wall backward is here plainly indicated, the vertical wall requiring nearly 50 per cent more material than the inclined one.

If the wall be of uniform thickness throughout, a equals b, and the solution of (59) gives

$$b = -C + \sqrt{C^2 + \frac{2P_1}{v}}, \quad . \quad . \quad . \quad . \quad . \quad (62)$$

in which C has the value

$$C = \frac{3h\cot\theta}{2} - \frac{2P_1\cos\theta}{vh}.$$

If in this θ be 90 degrees, it becomes

$$b = \sqrt{\frac{2P}{v}},$$

which is the proper thickness for a vertical rectangular wall. As an illustration take the same bank of sand as in the last example; then for $\theta = 80°$, $C = 4.81$ and the required thickness is $b = 4.0$ feet. If, however, $\theta = 90$ degrees, there is found

$b = 8.3$ feet. Here again the great advantage of inclining the wall is seen.

Sometimes it may be desirable to assume the inclination β of the front of the wall, and then to compute both b and a. For this case Figure 17 gives

$$a = b - h(\cot \beta - \cot \theta), \quad . \quad . \quad . \quad . \quad (63)$$

and inserting this in (59) and solving for b there is found

$$b = -D + \sqrt{D^2 + E}, \quad . \quad . \quad . \quad . \quad . \quad (64)$$

in which D and E are determined from

$$D = h(\cot \theta + \tfrac{1}{2} \cot \beta) - \frac{2P_1 \cos \theta}{vh},$$

$$E = \frac{2P_1}{v \sin \theta} + h^2(\cot^2 \beta - \cot^2 \theta).$$

For example, take the same bank of sand as before and let the back be vertical, or $\theta = 90°$, and $h = 20$ feet. Then $P_1 = 5650$ pounds per linear foot of wall. Now let the front of the wall have the batter of $1\frac{1}{2}$ inches per foot, or $\beta = 82° \ 52'$, and $\cot \beta = 0.1250$ (Article 13). Then $D = 1.25$ and $E = 68.5$ and from (64) the base thickness is $b = 7.12$ feet; lastly from (63) the top thickness is $a = 5.87$ feet.

The formulas above given can only be used when the earth pressure P_1 has a direction normal to the back of the wall. Those who believe in the theory of earth pressure set forth in Article 9 are referred to the latter part of Article 24 for a formula by which they should compute the thickness.

Problem 22. A wall weighing 140 pounds per cubic foot has a vertical back, is 18 feet high, and the horizontal earth pressure on it is 4580 pounds. Compute the thickness of the base when the cross-section is rectangular. Compute the thickness of the base when the cross-section is triangular. Compare the two sections with respect to amount of material.

ARTICLE 23. SECURITY AGAINST SLIDING.

(Checking sliding)

The base thickness b computed in the last Article provides proper security against the rotation of the wall under the lateral pressure of the earth. The cross-section thus determined should now be investigated and full security against sliding be provided. This can be done in three ways.

First: the masonry may be laid with random courses so that no through joints will exist. If the stones are of sufficient size this checks very effectually all liability to sliding.

Second: all through joints may be inclined backward at an angle α (Fig. 13) so that the resultant R shall be as nearly normal to them as possible. This will occur when F in formula (44) is zero, or when

$$P_1 \sin(\theta + \alpha) = V \sin \alpha,$$

and this reduces to

$$\cot \alpha = \frac{V}{P_1 \sin \theta} - \cot \theta, \quad \ldots \quad (65)$$

from which α can be computed for any joint, V being the weight of the wall above that joint, P_1 the earth pressure above it, and θ the inclination to the horizontal of the back of the wall. As b is computed for a horizontal base, the value of V is a little less than $\frac{1}{2}vh\ (a + b)$. For example, take the wall designed above where $\theta = 90$ degrees, $h = 20$ feet, $P_1 = 5650$ pounds, $v = 165$ pounds per cubic foot, $a = 5.9$ feet, and $b = 7.1$ feet. Then V is a little less than 21 450 pounds, say 21 000 pounds, and cot $\alpha = 2.7$, which gives $\alpha = 20$ degrees nearly. This backward inclination should be made less for joints above the base, becoming nearly zero for those near the top of the wall.

Third: for cases where a through horizontal joint cannot be avoided, as when a wall is built on a platform, the thickness of the base which will give a required factor of security against sliding can be computed from (43). To do this make both z and α equal to zero in (44), and substitute the values of F and N from (43), giving

$$nP_1 \sin\theta = f(V - P_1 \cos\theta).$$

Now in this let the value of V be inserted, namely,

$$V = \tfrac{1}{2}vh(a + b),$$

and the equation be solved for b, thus:

$$b = -a + \frac{2P_1(n \sin\theta + f\cos\theta)}{fvh}, \quad . \quad . \quad . \quad (66)$$

in which a is the top thickness, h the vertical height, θ the inclination of the back of the wall to the horizontal, P_1 the normal pressure of the earth, v the weight of the masonry per cubic unit, f the coefficient of friction of the masonry on the through horizontal joint, and b the base thickness for a factor of security of n against sliding, It would be desirable that n should be about 3.0, but to secure this the wall must be thicker than is required for rotation. Accordingly, this method of obtaining security against sliding should be used only when all other methods are impracticable. Thus in the last article a vertical rectangular wall is determined to be 8.34 feet thick when $h = 20$ feet and $P_1 = 5650$ pounds, $v = 165$ pounds per cubic foot; now, if $n = 3.0$ and $f = 0.5$, formula (66) gives $a = b = 10.3$ feet.

Problem 23. Compute the proper inclination of the joints in the rectangular wall of Problem 22 at distances of 6, 12 and 18 feet from the top.

ARTICLE 24. ECONOMIC PROPORTIONS.

By the help of the formulas of Article 22 the thicknesses of several trapezoidal walls will now be computed in order to compare the quantities of masonry required, and thus obtain knowledge regarding the most economical forms of cross-section. All the walls will be 18 feet in vertical height, and sustain a level bank of earth whose angle of repose is 34 degrees and which weighs 100 pounds per cubic foot. The weight of the masonry will be taken as 150 pounds per cubic foot.

Case I.—Let the back of the wall be inclined forward at a batter of two inches per foot, or $\theta = 99° 28'$ (Fig. 18). From formula (25) the normal earth pressure P_1 is found to be 5690 pounds. Then assuming the top thickness a at 0.0, 1.0, 2.0 feet, etc., the proper base thickness for each is computed from formula (60) and given in the table below.

Case II.—Let the back of the wall be vertical, as in Fig. 19, or $\theta = 90°$. From formula (26) the earth pressure P_1 is found

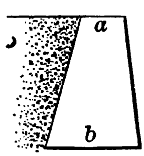


FIG. 18.

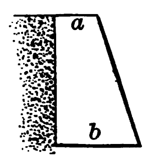


FIG. 19.

to be 4580 pounds. Then assuming thicknesses of the top, the corresponding base thicknesses are computed and inserted in the following table.

In this table the column headed "cubic yards" gives the quantity of masonry in one linear foot of the wall, and it is seen that in each case this is least for the wall with the thinnest top. It is also seen that the vertical walls require less masonry than the corresponding ones with forward batters. The columns headed "per cent" show these facts more clearly, the standard of comparison being the vertical rectangular wall which is taken as 100.

Assumed Top Thickness. *a*.	Case I. $\theta = 99° 28'$.			Case II. $\theta = 90°$.		
	Base Thickness. *b*.	Cubic Yards.	Per cent.	Base Thickness. *b*	Cubic Yards.	Per cent.
Feet.	Feet.			Feet.		
0.0	9.6	3.20	62	7.8	2.60	50
1.0	9.5	3.50	67	7.3	2.77	53
2.0	9.4	3.80	73	7.1	3.03	58
3.0	9.5	4.17	80	7.1	3.37	65
4.0	9.6	4.57	88	7.1	3.70	71
5.0	9.9	4.97	96	7.1	4.03	77½
6.0	10.2	5.40	104	7.2	4.40	85
7.0	10.5	5.83	112	7.5	4.83	93
7.8				7.8	5.20	100
7.9	10.9	6.27	120			

Case III.—Let the back of the wall be inclined backward at a batter of 1½ inches per foot, or $\theta = 82° 53'$ (Fig. 20). Here the earth pressure P_1 is found to be 3850 pounds. Then

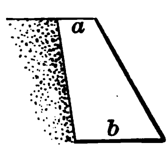


FIG. 20.

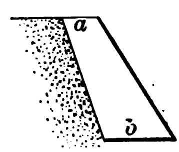


FIG. 21.

assuming values of the top thickness *a*, the corresponding values of the base thickness *b* are computed from (60) and given below in the tabulation.

Case IV.—Lastly, let the back of the wall be inclined still more backward, the batter being 3 inches per foot, or $\theta = 75° 58'$, as in Fig. 21. Then the earth pressure is found to be 3200 pounds, and as before values of b are computed for assumed values of a.

The following table gives the results of these computations for Cases III and IV, the columns "cubic yards" and "per cent" having the same signification as before. It is seen that the general laws of economy of material are the same, namely,

Assumed Top Thickness. a.	Case III. $\theta = 82° 53'$.			Case IV. $\theta = 75° 58'$.		
	Base Thickness. b.	Cubic Yards.	Per cent.	Base Thickness. b	Cubic Yards.	Per cent.
Feet.	Feet.			Feet.		
0.0	6.6	2.20	42	5.1	1.70	33
1.0	5.9	2.30	44	4.2	1.73	33
2.0	5.4	2.47	47½	3.4	1.80	35
2.9				2.9	1.93	37
3.0	5.1	2.70	52			
4.0	4.9	2.97	57			
4.9	4.9	3.30	63			

the thinner the top and the greater the backward batter of the wall the less is the quantity of masonry. The consideration of these principles in connection with the local circumstances of an actual case will hence tend toward economy of construction. Chief among these local circumstances is the price of land, and where this is very high a wall with a vertical front and a forward batter of back is often used, although this requires more masonry than any other form, for the saving in cost of the land may more than balance the extra expense for masonry. In all cases of design the first consideration is stability, and the second economy—not economy in the cost of material, but in the total expenditure of money.

Those who believe in the theory of earth pressure set forth in Article 9 may ask if its use would lead to the same conclusions regarding economic proportions. To decide this it is necessary to deduce a formula for the thickness of a trapezoidal wall under such pressure, and then to make the same computations for the four cases with the same data.

The fundamental formula (47) is good for all cases. In this let the values of p, V, and Vs be inserted from (49), (51) and (52), making $z = \phi$ and $t = \frac{1}{3}b$. Then results

$$P_2\left(2b\cos(\theta+\phi)+\frac{h\cos\phi}{\sin\theta}\right) = \tfrac{1}{3}vh(b^2+ab-a^2+bh\cot\theta+2ah\cot\theta),$$

and solving this with respect to b there is found

$$b = -A + \sqrt{B+A^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

in which the values of A and B are

$$A = \tfrac{1}{2}(a+h\cot\theta) - \frac{2P_2\cos(\theta+\phi)}{vh},$$

$$B = \frac{2P_2\cos\phi}{v\sin\theta} + a^2 - 2ah\cot\theta,$$

and from this the base thickness b can be computed for any values of the given data, namely, the angle of repose of the earth ϕ, its inclined pressure P_2 as found by Article 9, the angle of inclination of the back of the wall θ, the top thickness a, the vertical height h, and its weight per cubic unit v.

Using the same data, the inclined pressure P_2 has been computed for each case, and the base thicknesses found from formula (67) for the same assumed top thicknesses. The cubic yards in one linear foot of wall are next obtained, and an inspection of these shows that the same general laws hold as before, namely, the thinner the wall and the less the angle θ the less is the quantity of masonry required.

The subjoined table gives the quantities of masonry for Case I, Case II, and Case IV, and by comparing them with

Assumed Top Thickness. a.	Case I. $\theta = 99° 28'$.		Case II. $\theta = 90°$.		Case IV. $\theta' = 75° 58'$.	
	Cubic Yards.	Per cent Difference.	Cubic Yards.	Per cent Difference.	Cubic Yards.	Per cent. Difference.
Feet.						
0.0	2.43	24	1.77	32	1.38	19
1.0	2.77	21	2.00	28	1.40	19
2.0	3.10	19	2.30	24	1.45	19
3.0	3.50	16	2.63	22		
4.0	3.97	13	3.00	19		
5.0	4.50	9	3.40	15		
6.0	5.07	6				

those previously deduced it is seen that they are all less, the difference being greatest for the triangular walls and least for those of uniform thickness. The column "per cent difference" shows in each case the percentage of material which the walls designed under inclined pressure are less than the corresponding ones designed under normal pressure. As in practice walls are not built with a top thickness less than two feet, it may be said as a rough rule that the hypothesis of inclined earth pressure (Article 9) gives a wall from 10 to 20 per cent less in size than that of normal earth pressure (Article 8).

Problem 24. Deduce a formula for the thickness of a wall under inclined earth pressure when $a = b$. Compute the thickness and quantity of material of such a wall for Case I, for Case II, and for Case IV.

ARTICLE 25. THE LINE OF RESISTANCE.

Let a be the top thickness and b the base thickness of a trapezoidal wall whose height is h. Then the thickness b' at a vertical distance y below the top is

$$b' = a + (b - a)\frac{y}{h}, \quad . \quad . \quad . \quad . \quad . \qquad (68)$$

and this is represented by $B'C'$ in Figure 22. Let P' be the

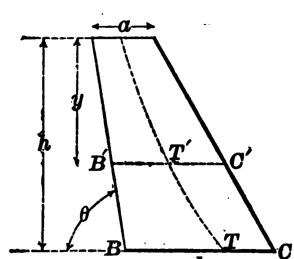


FIG. 22.

pressure of the earth, and V' the weight of the wall above $B'C'$. Let T' be the point where the resultant of P' and V'

cuts $B'C'$; as y varies T' describes a curve called the line of resistance. When y is zero T' coincides with the middle of the top. When y equals h the point T' coincides with T as determined by (50).

The line of resistance is the locus of the point of intersection of the resultant of the forces above any horizontal joint with the plane of that joint. This is a general definition applicable to triangular and curved sections as well as to trapezodial ones.

For a rectangular vertical wall under normal earth pressure the line of resistance is the common parabola. To prove this let the origin of coördinates be taken at the corner A in Figure 24, and let $AB' = y$ and $B'T' = x$. Now $P' = cy^2$, in which c

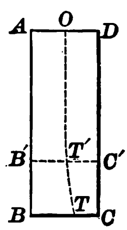


FIG. 23.

is a function of w, ϕ and δ (Article 8), and its lever-arm with respect to T' is $\frac{1}{3}y$. The value of V is vby, and its arm with respect to T' is $x - \frac{1}{2}b$. Then the equation of moments is

$$cy^2 \cdot \tfrac{1}{3}y = vby(x - \tfrac{1}{2}b),$$

or

$$x = \tfrac{1}{2}b + \frac{cy^2}{3vb},$$

which represents a parabola with its vertex at the middle of the top of the wall.

For a triangular wall with a vertical back the line of resistance is a straight line drawn from the top to the point where the resultant cuts the base. The proof of this is purposely omitted in order that it may be worked out by the student.

For a trapezoidal section the position of the line of resistance can be computed from (50), (51) and (52), making $z = 0$ for normal earth pressure, putting $P = cy^2$, $h = y$ and $b = b'$. For example, take a wall for which $\phi = 34°$, $\delta = 0°$, $w = 100$, $h = 18$, $\theta = 80°$, $v = 140$, $a = 2$ and $b = 5$ feet. Here from formula (25) P is found to be $11.02y^2$. From (68)

$$b' = 2 + \tfrac{1}{6}y,$$

and this inserted in (51) and (52) gives the values of V and Vs in terms of y. Then substituting all in (50) there is found

$$t = \frac{280 + 67.48y - 2.393y^2}{280 + 9.75y}.$$

From this equation the curve is now easily constructed, thus:

$y = 0$,	$t = 1.00$,	and	$b' = 2.00$
$y = 6$,	$t = 1.77$,	and	$b' = 3.00$
$y = 12$,	$t = 1.88$,	and	$b' = 4.00$
$y = 18$,	$t = 1.81$,	and	$b' = 5.00$

and it is seen that the line, while lying always within the middle third, departs most widely from the middle at the base of the wall.

Whatever be the form of cross-section the line of resistance can always be located by first determining the earth pressure and the weight of the wall for several values of y and then for each making a graphical construction as in Figure 12. The curve joining the points thus found on the several horizontal joints will be the line of resistance, and to insure proper stability against rotation it should lie within the middle third of the wall (Article 17).

Problem 25. Locate graphically the line of resistance in one of the walls of Case II, Article 24, determining points at depths of 6, 12 and 18 feet below the top.

ARTICLE 26. DESIGN OF A POLYGONAL SECTION.

Retaining walls with curved front are now and then built. The advantages claimed for such a profile are, first, finer architectural effect, and second, that the line of resistance may be made to run nearly parallel to the central line of the wall, thus making it a form of uniform strength and insuring economy of material.

The determination of the equation of a curved profile to fulfil the condition that the line of resistance shall cut every joint at the same fractional part of its length from the edge is of very great mathematical difficulty, if not impossibility, because the weight of the wall above any joint and its lever-arm are unknown functions of the coördinates of the unknown curve. By considering the curve to be made up of a number of straight lines, however, it is easy to arrange a profile to satisfy the imposed conditions which will not practically differ from the theoretical curve. The method of doing this will now be illustrated by a numerical example.

A wall 30 feet in vertical height is to be designed to support a level bank of earth whose angle of natural slope is 34 degrees and which weighs 100 pounds per cubic foot. The back of the wall is to be plane and to have an inclination of 80 degrees. The top of the wall is to be 2 feet thick, and the weight of the masonry is to be 165 pounds per cubic foot. It

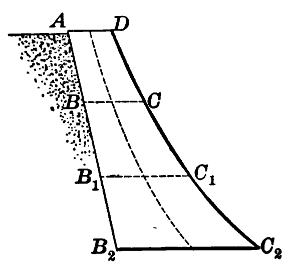


FIG. 24.

is required to design the wall so that the line of resistance shall cut the base B_2C_2 at its middle point, and also cut the lines B_1C_1 and BC at their middle points, B_1C_1 being 20 feet and BC 10 feet from the top. This insures a factor of infinity against rotation (Article 17) which is a greater degree of stability than is usually required in practice, but the method employed is general, and the example will serve to show how a wall may be designed to satisfy any imposed condition.

First, take the upper part $ABCD$ and consider it as a simple trapezoidal wall, upon which the normal earth pressure is found by (25) to be 1410 pounds. In the general formula (47) the values of p, V and Vs are now to be substituted from (49), (51) and (52), making $z = 0$ and putting t equal to $\frac{1}{3}b$. This gives an equation in which all quantities but b are known, and by its solution there is found the value $b = 4.47$ feet. This completely determines the cross-section $ABCD$ so that it is easy to find the weight $V = 5240$ pounds, and from (52) its lever-arm $s = 1.55$ feet.

Second, take the trapezoid BCC_1B_1 and consider it as acted upon by four forces, the weight of the upper part 5240 pounds, its own weight V, the normal pressure of 20 feet of earth

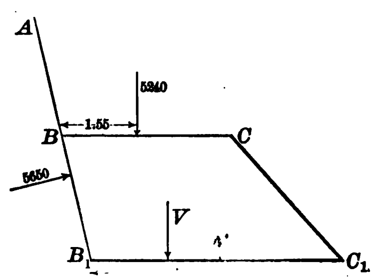


FIG. 25.

which is 5650 pounds acting at $6\frac{2}{3}$ feet vertically above B_1, and the reaction R of the wall below it which by the hypothesis passes through M, the middle point of B_1C_1. Let s be the lever-arm of V with respect to B, and let B_1C_1 be denoted by b. With respect to the centre M the lever-arm of V is

$\frac{1}{2}b - s$, that of the 5240 pounds is $\frac{1}{2}b + 0.21$, and that of the earth pressure is $6.77 + 0.087b$. Then the equation of moments is

$$5650(6.77 + 0.087b) = V(\tfrac{1}{2}b - s) + 5240(\tfrac{1}{2}b + 0.21).$$

Inserting in this the values of V and Vs in terms of b, and then solving, there is found $b = 8.70$ feet. This determines the cross-section so that its weight V is found to be 108.50 pounds, and the lever-arm of this with respect to B_1 to be 2.62 feet.

Lastly, the trapezoid $B_1C_1C_2B_2$ is treated in a similar manner, as acted upon by five forces, the weights 5240 and 10850 pounds, the pressure of 30 feet of earth which is 12 720 pounds applied at B_1, its own unknown weight V, and the reaction R which passes through the middle of B_2C_2. The lever-arms of the known forces with respect to that centre being found, the equation of moments is

$$12720(10.15 + 0.087b) = V(\tfrac{1}{2}b - s) + 5240(\tfrac{1}{2}b + 1.97) + 10850(\tfrac{1}{2}b - 0.86),$$

in which b is the base B_2C_2, and s is the lever-arm of V with respect to B_2. From (51) and (52) the values of V and Vs are to be expressed in terms of b and inserted; then by solution there is found $b = 13.6$ feet.

The points C, C_1 and C_2 in the profile of the cross-section are now known, and a curve may be drawn through them, or the front may be built with straight lines. The economy of the curved profile is indicated by the fact that the cross-section as determined is 209 square feet, whereas a trapezoidal section, designed under the same conditions has a base thickness of 15.2 feet and a cross-section of 257 square feet.

Problem 26. Design a curved wall for the same data as above, but under the condition that the line of resistance shall cut each of the bases BC, B_1C_1, B_2C_2 at one-third its length from the outer edge.

ARTICLE 27. DESIGN AND CONSTRUCTION.

When a retaining wall is to be designed its vertical height will be given. The inclination of its back and the thickness of its top are to be assumed, in accordance with the principles of Article 24, so as to result in the least total expenditure for land, labor and material. The form of section selected will be usually trapezoidal.

The normal earth pressure is now computed by the proper formula of Article 8.

The thickness at the base is then computed by formula (60), and thus the cross-section of a trapezoidal wall is determined. The batter of the front of the wall is known by (63), and if this proves to be greater or less than is thought advisable new proportions are assumed and another cross-section determined.

By the help of formula (65) the approximate inclination of a few of the joints should next be found so that the wall may be built with full security against sliding. It is not always necessary to give the joints the full inclination thus computed, however, since this implies a factor of security of infinity.

As a check on the computations it is well to make a graphical investigation of the proposed wall and determine the factors of security at the base against rotation (Article 18) and also against sliding (Article 15). These will, in general, be less for the base than for any joint above the base.

Lastly, the maximum pressure per square inch at the edge of the base joint may be computed (Article 20). If this is less

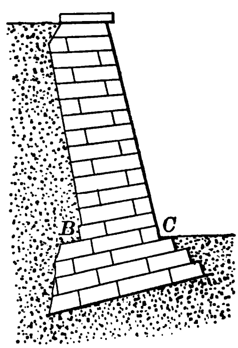


FIG. 26.

than the allowable working strength of the stone, the wall is safe against crushing. Only for very high walls will this computation be necessary.

The computed thickness b is the horizontal thickness of the wall at the top of the foundation, as BC in Figure 22. This foundation should be built with care, not only to bear the weight of the wall and prevent it from sliding, but also to protect it from the action of the rain and frost. Provision should be made for the drainage of the bank by longitudinal ditches and by weep-holes through the wall, so that water may not collect and increase the pressure.

It is good practice to batter the back of the wall slightly forward for about two feet near the top, in order that the frost may lift the earth upward without exerting lateral pressure against the wall.

Whether the wall be built with dry rubble or with cut stone in hydraulic mortar, great attention should be paid to details of workmanship and construction, all of which should be clearly set forth in the specifications. The earth must be thrown loosely against the wall or be dumped against it from above, but should be carefully packed in layers which slope upward toward the back.

Problem 27. Let $\phi = 38$ degrees, $\delta = 10$ degrees, $w = 100$ pounds per cubic foot, $v = 150$ pounds per cubic foot, $a = 2$ feet, $\theta = 80$ degrees. Compare the quantities of material required for two walls, one 9 feet high and the other 18 feet high.

CHAPTER V.

MASONRY DAMS.

ARTICLE 28. THE PRESSURE OF WATER.

All doubts regarding the direction and intensity of the lateral pressure against walls vanish when the earth is replaced by water. For since water has no angle of repose, $\phi = 0°$ and $\delta = 0°$, and all the formulas of Chapter II reduce to (35), which gives the normal water pressure against a wall of height h when the depth of the water is also h.

$P = \frac{1}{2} w \frac{h^2}{\sin\theta}$ $w = 62\frac{1}{2}$ lb.

The principles of hydrostatics show that the direction of water pressure is always normal to a submerged plane; also that the total normal pressure on such a surface is obtained by multiplying together the weight of a cubic unit of water, the area of the surface and the depth of its centre of gravity below the water level.

The water level is usually lower than the top of the dam, as shown in Figure 27. Let d be the vertical depth of the water above the base of a trapezoidal dam, θ the angle which the back makes with the horizontal, and w the weight of a cubic unit of water. Then the surface submerged is $\frac{d}{\sin\theta} \times 1$, the depth of its centre of gravity below the water level is $\frac{1}{2}d$, and hence the normal pressure is

$$P = \tfrac{1}{2}wd^2 \div \sin\theta,$$

which agrees with (35). The centre of pressure, or the point at which the resultant pressure must be applied to balance the actual presssures, is on the back of the dam at a vertical height

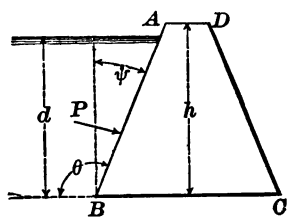


FIG. 27.

of $\frac{1}{3}d$ above the base; this is known by a theorem of hydro statics and likewise by Article 12.

The angle θ is never less than a right angle for masonry dams, and hence it will be convenient to use instead of it the angle ψ which the plane of the back makes with the vertical. Then $\theta = 90° + \psi$, and the normal pressure is

$$P = \tfrac{1}{2}wd^2 \sec\psi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

and for a vertical wall, where $\psi = 0°$, this becomes $P = \frac{1}{2}wd^2$.

The normal pressure P may be decomposed into a horizontal component P' and a vertical component P'', whose values are expressed by

$$P' = P\cos\psi = \tfrac{1}{2}wd^2, \qquad P'' = P\sin\psi = \tfrac{1}{2}wd^2\tan\psi\,; \qquad (70)$$

and if ψ be a small angle, as is usually the case, the horizontal component $\frac{1}{2}wd^2$ is sometimes taken as the actual water pressure. This is an error on the side of safety, since the vertical component, acting downward, increases the stability of the dam, unless the water penetrates under the base BC, which is an element of danger that ought not to be allowed.

Problem 28. For a waste-weir dam the water level may be higher than AD by an amount d_1. Prove that the normal pressure is

$$P = wh(\tfrac{1}{2}h + d_1) \div \cos\psi, \quad . \quad . \quad . \quad . \quad (71)$$

and that the centre of pressure is at a vertical distance d_0 above B, whose value is given by the formula

$$d_0 = \frac{h + 3d_1}{h + 2d_1}\cdot\frac{d}{3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

ARTICLE 29. PRINCIPLES AND METHODS.

The fundamental requirements concerning the design of masonry dams are the same as those governing all engineering work; first, stability, and second, economy. The first requires that the structure be built so that all its parts shall have proper strength, and the second that this shall be done with the least total expenditure of money. This expenditure consists of two parts, that for material and that for labor, and economy will result if material can be saved without increasing the labor. Hence all parts of a structure ought to be of equal strength (like the "one-hoss shay"), provided that the cost of the material thus saved is greater than the cost of the extra labor required; for if one part exceeds the others in strength it has an excess of material which might have been saved.

For ordinary retaining walls and for low masonry dams the trapezoidal form is the only practicable cross-section, since curved faces do not save sufficient material to balance the cost of the extra expense of construction. But for high masonry dams, and as such may be classed those over 80 or 100 feet high, it not only pays to deviate from the trapezoidal section, but it is often absolutely necessary to do so in order to reduce the pressure on the base to allowable limits. The section adopted in such cases is therefore an approximation to that of a form of uniform strength.

The general principles of stability of retaining walls set forth in the preceding pages apply to all masonry structures, but it will be well to state them briefly again, with especial reference to dams.

First, there must be proper stability against sliding at every joint and at every imaginary horizontal section. This can be done either by bonding the masonry with random courses so that no through joints exist, or by inclining such joints at the proper backward slope (Article 23). The first method is alone applicable to a dam, and by the use of hydraulic mortar the whole structure should be made monolithic.

Second, there must be proper stability against rotation at every horizontal section of the dam. This will be secured when the resultant of all the forces above that imaginary base cuts it within the middle third (Article 17) or at the most at the limit of the middle third. In a dam there will be two cases to be considered: (*a*) when the reservoir is full of water, and (*b*) when the reservoir is empty. For the first case the line of resistance should not pass without the middle third on the front or down-stream side, and for the second case it should not pass without it on the back or up-stream side.

Third, there must be proper security against crushing at every point within the masonry. As a general rule this demands that the compressive stress per square inch shall not exceed 150 pounds, although in a few cases higher values have been allowed.

It will be found in designing a high dam that the second principle will determine the thicknesses for about 100 feet below the top. For greater heights the third principle must generally be used, and the formulas of Article 20 be applied. It is indeed doubtful whether these formulas correctly represent the actual distribution of stress on the base of a high dam with a polygonal cross-section, for it would naturally be thought that greater stresses would obtain near the middle rather than near the edge of the base. If such is the case, however, the application of the formulas can only err on the side of safety.

Problem 29. A masonry dam 36 feet high and 24 feet wide weighs 150 pounds per cubic foot. Find the point where the resultant cuts the base when the water is 33 feet deep above the base.

ARTICLE 30. INVESTIGATION OF A TRAPEZOIDAL DAM.

[[illegible] designed or completed].

The given data will furnish the dimensions of the dam, and the normal water pressure on its back will be computed by (69). Then by the method of Article 18 a graphical investigation for rotation may be made and the factor of security be determined for any joint BC. Through joints should not exist in a masonry dam, and hence BC will be taken as horizontal in the construction, or even if they do exist BC may be an imaginary horizontal joint.

The factor of security against rotation may be computed by the formulas of Article 19, first making $z = 0°$, and $\theta = 90° + \psi$. Then from the given data P is found by (69), V and Vs by (51) and (52), and t is computed by (50), in which h is to be put equal to d, whence finally n is derived by (48). It will however be more satisfactory for a student to make an analysis directly from first principles rather than to arbitrarily use formulas for mere computation. This will be now done for a particular example.

The largest trapezoidal dam is that at San Mateo, California. The top thickness is 20 feet, the base thickness is 176 feet, the vertical height is 170 feet, the batter of the back is 1 to 4, and the masonry is concrete, which probably weighs about 150 pounds per cubic foot. It is required to investigate its stability when the water is 165 feet deep above the base.

Let P be the normal water pressure on the back, and V the weight of dam, both per foot of length. Let BC be the base, and T the point where the resultant of P and V cuts it.

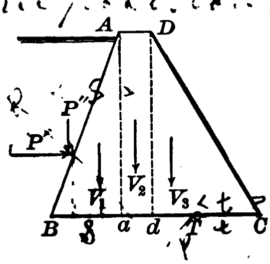


FIG. 28.

Now with respect to this point the moment of P will equal the moment of V. Let p be the lever-arm of P; let t represent the distance CT, and s the horizontal distance from B to the line of direction of V. The lever-arm of V is then $b - s - t$, and the equation of moments is

$$Pp = V(b - s - t) \quad \text{or} \quad Pp = Vb - Vs - Vt. \quad (73)$$

The first member of this equation may be replaced by $P'p' - P''p''$, in which P' and P'' are the horizontal and vertical components of P, and p' and p'' are their lever-arms with respect to the point T. Also V may be replaced by vA, where v denotes the weight of the masonry per cubic foot and A is the area of the cross-section. Then

$$P'p' - P''p'' = v(Ab - As - At), \quad . \quad . \quad . \quad (74)$$

which is a formula better adapted to numerical operations.

To apply this to the San Mateo dam the data are $d = 165$ feet, $\tan\psi = 0.25$, $a = 20$ feet, $b = 176$ feet, $h = 170$ feet, and $v = 150$ pounds per cubic foot. Then from (70)

$$P' = 850\,780 \text{ pounds}, \quad P'' = 0.25P = 212\,700 \text{ pounds},$$

and from the figure,

$$p' = \tfrac{1}{3}d = 55 \text{ feet}, \qquad p'' = 176 - 0.25 \times 55 - t.$$

Also the area of the trapezoid is

$$A = \tfrac{1}{2} \times 170(176 + 20) = 16\,660 \text{ square feet},$$

and the moment As is computed by regarding A as the sum of the triangles AaB and DdC and the rectangle $AadD$ (Figure 28), thus: or saying that the total moment of T around B = sum of individual moments

$$As = AaB \times \tfrac{2}{3}Ba + AadD(Ba + \tfrac{1}{2}ad) + DdC(BC - \tfrac{2}{3}dC),$$

around same point.

whence $As = 1\,248\,820$ ~~feet cube~~ cu. ft. Inserting now all values in (74) and solving for t there is found $t = 88.6$ feet. The resultant therefore cuts the base very near the middle, so that the factor of security against rotation is practically infinity (Article 17).

It is the custom of some engineers to neglect the vertical component of the water pressure, and regard only the horizontal component. Testing the San Mateo dam under this supposition, P'' equals zero, and, all other quantities being the same as before, there is found $t = 80.2$ feet, whence the factor of security against rotation is

$$n = \frac{88}{88 - 80.2} = 11.3,$$

which shows that the degree of stability is ample.

A masonry dam should be investigated not only for the case when the reservoir is filled with water, but also for the case when the reservoir is emptied. Here the tendency to rotation, or overturning, is usually backward instead of forward. Let S be the point where the direction of the weight V cuts the base, and let M be the middle of the base. Then the factor of security is the ratio of MB to MS, or

$$n = \frac{\frac{1}{2}b}{\frac{1}{2}b - s},$$

in which the distance s is computed by dividing the value of As by that of A. Now, for the San Mateo dam,

$$s = \frac{1\,248\,820}{16\,660} = 75.0 \text{ feet},$$

and then n is found to have a value of 6.8.

No through joints exist in this dam, and the method of construction of the base is such as to preclude all possibility of sliding. Moreover by the use of (43) the coefficient of friction which will allow sliding to occur on the base is

$$f = \frac{850\,780}{150 \times 16\,660} = 0.34,$$

a value which would be very low for an imperfect construction.

The compressive stresses on the base may next be investigated by the method of Article 20. When the water in the reservoir is 165 feet deep the resultant R cuts the base so near the middle that the compression can be regarded as uniformly distributed. The pressure normal to the base is $V + P''$, and hence the stress per square inch is

$$S = \frac{150 \times 16\,660 + 212\,700}{144 \times 176} = 107 \text{ pounds},$$

which is probably less than one-sixteenth of the ultimate strength of good concrete when one year old.

If the reservoir should be empty the greatest stress would come at the heel B, and as V is applied at 75 feet from B, that stress in pounds per square inch is, from formula (58),

$$S = \frac{2 \times 150 \times 16\,660}{144 \times 176}\left(2 - \frac{3 \times 75}{176}\right) = 142.$$

It will also be found that the stress at the middle of the base is 99 pounds per square inch, that at the toe C is $142 - 99 = 43$ pounds per square inch.

Problem 30. Investigate the security of the San Mateo dam for a horizontal section 100 feet below its top—(*a*) when the water is 95 feet deep above that section; (*b*) when the reservoir is empty.

ARTICLE 31. DESIGN OF A LOW TRAPEZOIDAL SECTION.

When a trapezoidal dam is to be designed its height *h* will be given, and also the depth *d* of the water behind it. The weight per cubic foot of the masonry *v* will be known, at least approximately. The thickness of the top, *a*, will be assumed; usually this will serve for a roadway or footway and hence cannot be less than 8 or 10 feet. The batter of the back, or

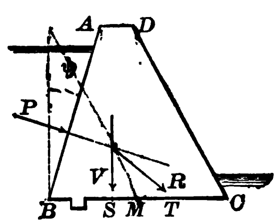
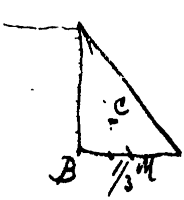


FIG. 29.

tan ψ, is next assumed, and usually this will be taken small in order that the weight of the wall *V* may fall as far away from the toe *C* as possible. Let *M* be the middle of the base *BC*; let *S* be the point where the direction of *V* cuts it, and *T* the point where the direction of the resultant *R* cuts it. It is plain that *MS* will always be less than one-third of *MB* for any trapezoid whose back leans forward, and that it becomes equal to one-third of *MB* only when *AD* is zero and *AB* is vertical.



Let b be the length of the base BC, and let t be the distance CT. It is required to find b so that MT shall be one-third of MC, or, what is the same thing, that t shall equal $\frac{1}{3}b$. Full security against rotation will then exist both for reservoir full and for reservoir empty.

Formula (74) is a fundamental one applicable to any section. To apply it to the problem in hand, the values of the lever-arms l' and l'' are to be stated in terms of the other quantities, thus:

$$l' = \tfrac{1}{3}d, \qquad l'' = b - \tfrac{1}{3}d \tan\psi - t. \quad . \quad . \quad (75)$$

Also the area A is expressed by

$$A = \tfrac{1}{2}h(a + b), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

and by the method of the last Article the value of the moment As is found to be

$$As = \tfrac{1}{6}h\Big(a^2 + ab + b^2 + h(2a + b)\tan\psi\Big). \quad . \quad (77)$$

Inserting, now, all these quantities in (74), and making $t = \frac{1}{3}b$, there is found a quadratic equation in b whose solution gives

$$b = -F + \sqrt{F^2 + G}, \quad . \quad . \quad . \quad . \quad . \quad (78)$$

in which F and G have the values

$$F = \frac{1}{2}\Big(\frac{4P''}{vh} + a - h\tan\psi\Big),$$

$$G = \frac{2d}{vh}(P' + P''\tan\psi) + a^2 + 2ah\tan\psi,$$

and from these the proper base thickness can be found, P' and P'' being first computed by (70), or if desired the expressions for F and G can be written

$$F = \frac{1}{2}\left(\frac{2d^2 \tan\psi}{gh} + a - h\tan\psi\right),$$

$$G = \frac{d^2 \sec^2\psi}{gh} + a^2 + 2ah\tan\psi;$$

in which g is the ratio of v to w, or the specific gravity of the masonry.

If $\psi = 0°$, the formula (78) takes the simple form

$$b = -\tfrac{1}{2}a + \sqrt{\frac{d^2}{gh} + \frac{5}{4}a^2}, \quad . \quad . \quad . \quad . \quad (79)$$

which gives the proper base thickness of a trapezoidal dam with a vertical back.

The compressive stress at C in pounds per square inch is now found from (55), or

$$S = \frac{2(V + P'')}{144b} = \frac{vA + \tfrac{1}{2}wd^2 \tan\psi}{72b}; \quad . \quad . \quad (80)$$

and if this is less than the specified limiting value, no further investigation will be necessary; but if greater, then the above formulas for thickness will not apply and those of the next Article must be used. The limiting value of S is often taken at 150 pounds per square inch.

The compression at the inner edge B when the reservoir is empty is less than that at C when it is full, for in any trapezoid where $\tan\psi$ is positive MS is less than one-third of MB. The distance BS can, however, be obtained by dividing (77) by (76), whence

$$s = \frac{a^2 + ab + b^2 + (2a + b)\,h\tan\psi}{3(a + b)}, \quad . \quad . \quad . \quad (81)$$

and then by the use of (58) the unit-stress at B is computed.

In order to show the application of the formulas and at the same time study the question of economic proportions, let the following data be taken: $h = 60$ feet, $d = 57$ feet, $a = 9$ feet, $v = 150$ pounds per cubic foot or $g = 2.4$. Let three designs be made for which the back has different batters, namely, $\tan\psi = \frac{1}{6}$, $\tan\psi = \frac{1}{12}$, and $\tan\psi = 0$. Using the formula (78), the base is first found, and then by (76) the area of each trapezoid; thus:

$\tan\psi = \frac{1}{6}$,	$b = 36.5$ feet,	$A = 1365$ sq. ft.,	$= 109$ per cent
$\tan\psi = \frac{1}{12}$,	$b = 34.4$ feet,	$A = 1302$ sq. ft.,	$= 104$ per cent
$\tan\psi = 0$,	$b = 32.75$ feet,	$A = 1253$ sq. ft.,	$= 100$ per cent

From which it is seen that the most advantageous section is the one with the vertical back. This conclusion might also be inferred from the discussion in Article 24.

It is the custom of some engineers to neglect the vertical component of the water pressure. Formula (78) may be adapted to this hypothesis by making P'' equal to zero in the quantities F and G, which then become

$$F = \tfrac{1}{2}(a - h\tan\psi),$$

$$G = \frac{d^3}{gh} + a^2 + 2ah\tan\psi.$$

The thickness of the dam computed under this hypothesis is greater than before. Thus, for the above example,

$\tan\psi = \frac{1}{6}$,	$b = 39.8$ feet,	$A = 1466$ sq. ft.,	$= 117$ per cent
$\tan\psi = \frac{1}{12}$,	$b = 36.2$ feet,	$A = 1356$ sq. ft.,	$= 108$ per cent
$\tan\psi = 0$,	$b = 32.75$ feet,	$A = 1253$ sq. ft.,	$= 100$ per cent

Problem 31. Find the compressive unit-stress at B and C for one of the cases of the above numerical example.

ARTICLE 32. DESIGN OF A HIGH TRAPEZOIDAL SECTION.

When the value of h is so great that the formula for thickness deduced in the last article cannot be used the dam is said to be "high." For such cases the condition $t = \frac{1}{3}b$ cannot be applied, but t must be made greater than $\frac{1}{3}b$ so as to reduce the unit-stress at the toe C. The base thickness will hence be greater than that given by (78).

Let S be the given limiting unit-stress in pounds per square foot. The corresponding value of t is, from (57),

$$t = \tfrac{2}{3}b - \frac{b^2 S}{6(vA + P'')}, \quad \ldots \ldots \quad (82)$$

in which vA is the equivalent of the weight V. Inserting this in (74), and also the values for p', p'', A and As, there is deduced a quadratic in b whose solution gives

$$b = -K + \sqrt{K^2 + L}, \quad \ldots \ldots \quad (83)$$

in which K and L have the values

$$K = (P'' - \tfrac{1}{2}vh^2 \tan\psi)\frac{1}{S},$$

$$L = \left(2d\,(P' + P''\tan\psi) + vh(a^2 + 2ah\tan\psi)\right)\frac{1}{S}.$$

If in these $P'' = 0$, the vertical component of the water pressure is neglected; and if $\tan\psi = 0$, the back of the trapezoid is vertical.

In using these formulas the given data are a, h, d, $\tan\psi$, v and S. Then b is computed, taking the water pressures P' and P'' from (75). When b is found, s should be determined by (81), and then by (58) the stress at B when the reservoir is empty is computed.

For an example take $a = 20$ feet, $h = 170$ feet, $d = 165$ feet, $\tan\psi = 0.2$, $v = 150$ pounds per cubic foot and $S = 21\,000$ pounds per cubic foot. Let it be required to find b, neglecting the vertical component P'' of the water pressure. From Article 28 the value of P' is 850 780 pounds, and by hypothesis $P'' = 0$. Then inserting all values, $K = -20.64$, $L = 15\,506.6$, whence $b = 145.2$ feet. This gives for the area of the section $A = 14\,042$ square feet, and from (82) $t = 0.425\,b$, which locates the point where the resultant pierces the base when the reservoir is full. From (81) there is found $s = 61.9$ feet $= 0.426b$, which gives the point where the line of action of V cuts the base, and when the reservoir is empty the unit-stress at the back edge of the base is, by (58),

$$S_1 = \frac{2 \times 150 \times 14042}{145.2}(2 - 3 \times 0.426) = 21\,000 \text{ nearly},$$

so that the compression at B for reservoir empty is about the same as that at C for reservoir full.

Problem 32. Discuss the above example without neglecting the vertical component of the water pressure.

ARTICLE 33. ECONOMIC SECTIONS FOR HIGH DAMS.

A high trapezoidal dam designed so as to give proper security against crushing on the base has an excess of stability in its upper part. Accordingly if the section be polygonal, or bounded by curved lines, both in front and back, these may be arranged so as to save material in the upper parts, thus lessening the weight that comes on the base, and hence reducing its width from that which a trapezoidal section would require. Such a structure will be approximately one of uniform security against rotation in its upper portions, and of uniform security against crushing in its lower portions. The method of designing the upper part will be similar to that used in Article 26 for the retaining wall.

Local and practical considerations will determine the thickness of the top AD. From the principles deduced in Articles 24 and 31 it is plain that to secure the greatest economy of material the back should be vertical for some distance below the top. If the upper sub-section $AA'D'D$ be rectangular, the line of resistance for the case of reservoir empty will cut the middle of $A'D'$; and if the height be properly chosen, the line of resistance for reservoir full will cut it at the front edge of the middle third. To find what this height should be let a be the thickness, h' the height AA', and d the depth of water

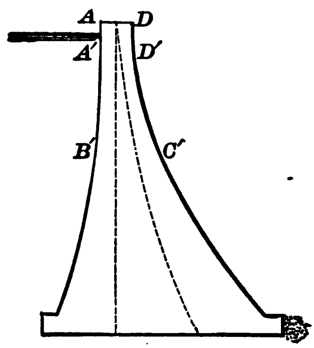


FIG. 30.

above A'. Then the equation of moments with reference to a point in the base distant $\frac{1}{2}a$ from D' is

$$\tfrac{1}{2}wd^2 \times \tfrac{1}{3}d = vah' \times \tfrac{1}{6}a.$$

Now if d be taken equal to h', as it may be in an extreme case, the solution of this gives

$$h' = a\sqrt{g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

in which g is the ratio of v to w, or the specific gravity of the masonry.

The next sub-section should be a trapezoid, and the entire section in fact may be considered as made up of trapezoids, the widths of these being so determined as to secure economy and stability. The former requires that the back should be vertical or that its batter should be as small as possible, and

Because 150 lbs ... a ...
height of $\frac{180 \text{ lbs}}{1728''} \times h' \times 12'' = 150$ lbs
i.e. h = 120 feet, in a cylinder, or prism.

the latter requires that the lines of resistance for reservoir full and reservoir empty shall not pass without the middle third, while the resulting unit-stresses are kept within the specified limit.

In the upper part of the dam the question of the compression of the masonry need not be considered, and the width of the base of each sub-section will be found from the requirement that the line of resistance for reservoir full cuts that base at one-third the length from the front edge.

In the lower part of the dam the widths are to be determined by regarding the compressive stresses. Owing to uncertainties concerning the theory of distribution of these stresses, and to differences of opinion concerning the manner in which it should be applied, engineers have not agreed upon a uniform method of design. The general form of section, however, is that shown in Figure 30, the back being battered below a certain depth in order to keep the line of resistance for reservoir empty well within each base, while the batter of the front increases downward. The views of different authorities are fully set forth in WEGMANN'S Design and Construction of Masonry Dams (second edition, New York, 1889), where also are given sectional drawings of all existing high dams.

Problem 33. Prove that a triangular section is one of uniform stability against rotation when the water level is at the vertex of the triangle.

ARTICLE 34. INVESTIGATION OF A POLYGONAL SECTION.

The graphical investigation of the stability and security of a polygonal section like Figure 30 is so simple in theory that space need not here be taken to set it forth in detail. The general method of Article 18 is to be followed for the base of each sub-section, and the only difficulty that need to occur will be in connection with determining the positions of the centres of gravity of the areas above the successive bases. These may be best computed by the method explained below. When the points S and T have been found for each base the factor of security against rotation is known by Article 17, both for reservoir full and reservoir empty, and then the maximum compressive stresses are determined as in Article 20.

The analytical investigation begins with the top sub-section, which is either a rectangle or a trapezoid (Figure 30), and finds as in Article 30, or by the formulas of Article 19, the degree of security for its base $A'D'$. Thus is determined the area A_1, the corresponding weight V_1, and the horizontal distance s_1 from its point of application to its back edge. Now let $A'BCD'$ be the next trapezoid, let h be its vertical height, a its top width, b its base width, ψ the angle of inclination of the back to the vertical, A_2 its area, v the weight of the masonry per cubic unit, V_2 its weight vA_2 which is applied at a horizontal distance s_2 from the back edge B. The sum $A_1 + A_2$ is the total area A whose weight is $vA = V$, and the line of action of this cuts the base at S, whose horizontal distance from the

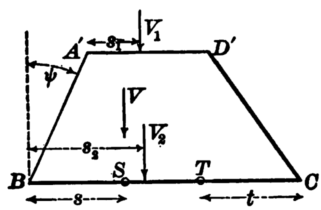


FIG. 31.

back edge B is called s. The value of s can be obtained by taking moments about B, thus:

$$s = \frac{A_1(s_1 + h \tan \psi) + A_2 s_2}{A_1 + A_2}, \quad . \quad . \quad . \quad (85)$$

which is the formula for locating the line of resistance when the reservoïr is empty. The values of A_2 and $A_2 s_2$ are found from the given quantities a, b, h, $\tan \psi$ by the help of (76) and (77).

When the reservoir is full let P' be the horizontal component of the water pressure on the entire back above B, and P'' the vertical component. Let their lever-arms with respect to T be p' and p''. Then the equation of moments is

$$P'p' - P''p'' = v(A_1 + A_2)(b - s - t). \quad . \quad . \quad (86)$$

In this the value of P' is $\frac{1}{2}wd^2$, and that of p' is $\frac{1}{3}d$. If the batter of the back be uniform from the top to B, the values of P'' and p'' are known by (70) and (75). If, however, the different trapezoids have different batters, values for P'' and p''

are not easily expressed. Hence it is often customary to neglect P'', and then the distance CT is

$$t = \frac{d^2}{6g(A_1 + A_2)(b - s)}, \quad . \quad . \quad . \quad . \quad . \quad (87)$$

in which g is the specific gravity of the masonry. From this the line of resistance can be located when the reservoir is full.

The factor of security against rotation can now be found, if desired, by (45) both for the case of reservoir empty and that of reservoir full. The degree of security against crushing will be deduced by computing the unit-stresses at B and C by the help of the formulas of Article 20 and then comparing these with allowable and with ultimate values. The degree of security against sliding could be easily determined if the coefficient of friction were known, but as the base is not a real joint, it will be sufficient to use formula (39), and deduce the value of f which would allow motion if a joint actually existed.

The above formulas can be applied to each trapezoid in succession, A_1 being taken as all the area above its top, and thus the lines of resistance can be traced throughout the entire section.

As a numerical example let it be required to test the fourth trapezoid of the theoretical section of the Quaker Bridge Dam given in Article 35. The data are $A_1 = 1823$ square feet, $s_1 = 12.4$ feet, $h = 20$ feet, $\tan \psi = 0.115$, $a = 37.4$ feet, $b = 53.4$

feet, $d = 90$ feet, and $g = \frac{v}{w} = 2\frac{1}{2}$; and it is required to find s and t with the unit-stresses S_1 and S. First the area of the given trapezoid is 908 square feet, and its moment $A_2 s_2$ is 21 808 feet cube. Then from (85) the value of s is 17.8 feet, and inserting this in (87) there is found $t = 17.8$ feet. The lines of resistance here cut the base at the ends of the middle third so that the factors of security for reservoir full and for reservoir empty are each 3.0 (Article 17). The unit-stresses S_1 and S are also equal, and each will be found to be 111 pounds per square inch. Lastly, from (1) or (39) the coefficient of friction necessary for equilibrium is 0.59, a value which cannot be approached in a monolithic structure.

Some authors use the term "factor of safety" as meaning the ratio of the horizontal water pressure which would cause overturning to the actual existing horizontal water pressure. This should not be confounded with the factor of security used in this book.

Problem 34. Given a, b, h, V_1 and s_1 for any trapezoid (Figure 31). Deduce the value of tan ψ so that s shall equal $\frac{1}{3}b$.

ARTICLE 35. DESIGN OF A HIGH ECONOMIC SECTION.

The application of formula (86) will in general lead to complicated equations, unless the vertical component of the water pressure P'' is neglected. This is an error on the side of safety and is hence often allowable, particularly when tan ψ is small.

The following method is essentially like that devised by WEGMANN for the design of the Quaker Bridge Dam, and is here given because of all the different methods it appears to be best adapted to the comprehension of students.

Using the same notation as in the last article, the top width a is first assumed, and the uppermost sub-section is made a rectangle whose height is by (84) equal to $a\sqrt{g}$. The following sub-sections will be trapezoids with vertical backs, each base being determined so that $t = \frac{1}{3}b$. To find b for any trapezoid there will be given A_1 and s_1 from the preceding trapezoids, its upper base a, its height h, the total depth of water d, and the specific gravity g, while tan ψ equals zero. First A_2 and A_2s_2 are expressed in terms of a, b and h, by (76) and (77), and these are inserted in (85). Then the resulting expression for s is put into (86) and t made equal to $\frac{1}{3}b$. Thus is obtained a quadratic, whose solution gives

$$b = -K + \sqrt{K^2 + L}, \quad . \quad . \quad . \quad . \quad . \quad (88)$$

in which K and L have the values

$$K = \frac{1}{2}\left(\frac{4A_1}{h} + a\right), \qquad L = \frac{d^3}{gh} + \frac{6A_1s_1}{h} + a^2.$$

If, in these, A equals zero, the formula reduces to (79), which should be the case, as the whole section above the base then becomes a single trapezoid.

After having found the base of a trapezoid by (88) the value of s should be computed by (85), taking $\tan \psi = 0$. This will be at first greater than $\frac{1}{3}b$, but in descending lower (usually before d becomes 100 feet) a trapezoid will be found where s exceeds $\frac{1}{3}b$. As soon as this occurs formula (88) ceases to be applicable, for the section has not a sufficient degree of stability when the reservoir is empty. The back must now be battered so that s shall equal $\frac{1}{3}b$, at the same time keeping $t = \frac{1}{3}b$. Introducing these two conditions into (86) and solving for b there results

$$b = -\left(\frac{a}{2} + \frac{A_1}{h}\right) + \sqrt{\frac{d^2}{9h} + \left(\frac{a}{2} + \frac{A_1}{h}\right)^2}, \quad . \quad . \quad (89)$$

which gives the base of the trapezoid, and thus A_2 becomes known. The amount of batter required is now found by inserting in (85) the value of s_2 from (81), and solving for $\tan \psi$, namely:

$$\tan \psi = \frac{A_1(s - s_1) + A_2(s - \frac{1}{3}h) - \frac{1}{6}a^2h}{(A_1 + \frac{1}{3}A_2 + \frac{1}{6}ah)h}, \quad . \quad . \quad (90)$$

in which s is to be taken as $\frac{1}{3}b$. Thus the trapezoid is fully determined, and the next one can be designed, taking $A_1 + A_2$ as the new A_1, s as the new s_1, and b as the new a.

After having found the base of a trapezoid by (89), the compressive unit-stress at the ends of said base should be computed by (80). The value of this will be at first less than the allowable limit, but in descending lower (usually before d becomes 150 feet) a trapezoid is reached where it is greater. As soon as this occurs formula (89) ceases to be applicable, for the base of the section has not sufficient security against crushing.

The next value of b is to be derived by taking t as given by (82) and making $s = \frac{1}{3}b$. These introduced into (86) produce a quadratic in b, and this will be used until the compressive stress at B reaches the allowable limit. When this occurs s must be made greater than $\frac{1}{3}b$ by expressing its value from (58) in a manner analogous to (82). The two values of s and t are thus stated in terms of S_1 and S, the limiting unit-stresses at B and C, and inserting them in (86) and solving for b a quadratic is found from which all the remaining trapezoids are computed. As soon as any b is found A_2 is known, and then s is derived by (85), taking $A_1 + A_2$ as A. Lastly, using this value of s, the batter tan ψ is derived by (90).

This method is open to the objection that the formulas of Article 20 do not probably give the correct law of distribution of stress on the base of polygonal sections, and also to the objection that the water pressure is always taken as horizontal in direction. On the other hand, it has the advantage of being simple in use, whereas other methods but little, if any, more accurate in principle lead to equations of high degree whose solution can only be effected by tentative processes.

By the help of this method the engineers of the Aqueduct Commission of the city of New York deduced an economic section for the proposed Quaker Bridge Dam. The top thickness was taken at 20 feet and the specific gravity of the masonry at 2.5. The following are results for the theoretical section to a depth of 171 feet (see Table II in Report of the Aqueduct Commission, 1889).

d	b	A	tan ψ	t	$s_{\prime}$	S	S_1
34.7	20.0	834	0	6.7	10.0	13 031	6 516
50	26.2	1187	0	8.7	10.5	14 156	11 328
70	37.4	1823	0	12.5	12.4	15 234	15 234
90	53.4	2731	0.115	17.8	17.8	15 984	15 984
110	71.2	3977	0.100	25.2	23.7	16 391	17 453
130	92.9	5618	0.170	35.1	31.7	16 384	18 462
150	114.6	7698	0.170	45.3	40.1	17 078	19 930
171	137.4	10339	0.171	56.1	49.1	18 219	21 822

In this table the first column contains the depth of the water in feet, the second the base of each sub-trapezoid in feet, the third the total area above that base in square feet, the fourth the batter of the back, the fifth and sixth the distances in feet from the front and back edges of the base to the lines of resistance, and the seventh and eighth the stresses at those edges in pounds per square foot. It will be seen that the San Mateo dam, 170 feet high (Article 30), has about 61 per cent more material than this economic section of 171 feet height.

Problem 35. Design an economic section, taking the top thickness as 30 feet and the specific gravity of the masonry as $2\frac{1}{3}$.

ARTICLE 36. ADDITIONAL DATA AND METHODS.

There has now been given snch a presentation of the theory of masonry dams, adapted to the needs of students, as will serve to exemplify the principles which govern their design. A few concluding remarks concerning data, principles, and methods will now be made.

The force of the wind has not been considered in the data. If the wind blows up-stream when the reservoir is filled, the stability of the dam is increased; if it blows down-stream, its effect will be to produce waves rather than to add to the water pressure on the back.

The pressure due to the impulse of waves may be inferred from the fact that the highest pressure observed by STEVENSON in his experiments was 6100 pounds per square foot. The maximum horizontal pressure per linear foot on the top of a dam from wave action can, therefore, probably not exceed this value acting over three or four feet of vertical depth, and this only when the reservoir is of wide extent.

The horizontal pressure at the water line due to the thrust of ice should be taken, in the opinion of a board of experts on the Quaker Bridge Dam, to be 43 000 pounds per linear foot. (Report of the Aqueduct Commission, 1889.)

Let H be the horizontal force at the water line due to ice thrust, or wave action. Its moment will be Hd, and this is to be added to the moment of the water pressure. In all the preceding formulas, therefore, the quantity $\frac{d^2}{gh}$ should be replaced by $\frac{d^2}{hg} + \frac{6Hd}{vh}$ in order to include the effect of this horizontal force in the computations. For instance, if the example in Article 32 is to include the effect of the ice thrust, formula (84) must be modified as stated, taking $H = 43\,000$ pounds. Then b will be found to be 156.3 feet instead of 146.8, and the area of the trapezoid will be about $5\frac{1}{2}$ per cent greater than before.

When the computations extend below a permanent water level on the front of the dam the effect of the back pressure can easily be introduced into the formulas by substituting $d^2 - d_1^2$, for d^2, where d is the depth of the water on the back of the dam, and d_1 that on the front.

When the back and front of the dam are covered with earth or gravel below a certain level its action may be approximately estimated by computing the earth pressures according to the method of Article 8, and then adding the moments of these to the other external moments. Such computations however, will always be liable to more or less uncertainty, and hence should be made with caution.

It is not probable that the theory of Article 20 gives the correct distribution of stress on the wide base of a polygonal section, and it seems more likely that in such cases the unit-pressures at the ends of the base are less than those near the middle. If this is the case, the formulas probably err on the side of safety, even though they neglect the influence of the shearing stress due to the horizontal pressures. It is known (see Mechanics of Materials, Article 75) that a shear combines with a compression normal to it and produces in another direction a greater compression. But the application of this principle to stresses in masonry can scarcely be made until experimental evidence is afforded concerning the laws of distribution of the unit-stresses.

The theory of a dam which is curved in plan and which acts more or less like an arch has not been considered here. It may be stated as the general consensus of opinion, that a section which resists water pressure by gravity alone, like those designed in these pages, will not usually be rendered stronger by being curved in plan. A curve, however, is pleasing to the eye and impresses the observer with an idea of strength, so that it is often advisable to employ it, even if the length of the dam be slightly increased.